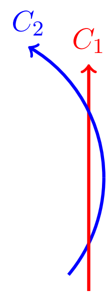
Entangled Polymers

In this problem, we will show a remarkable connection between the theory of entangled polymers, and Chern-Simons gauge theory.



Let's begin by deriving a topological fact about two entwined closed curves, C_1 and C_2, in the plane $\mathbb{R}^3$, as depicted in the picture above. Almost 200 years ago, Gauss showed that the **linking number** defined by the integral

$$L \equiv \frac{1}{4\pi} \oint_{C_1} \mathrm{d}\mathbf{x}_1 \times \oint_{C_2} \mathrm{d}x_2 \cdot \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3}$$

is a topological invariant of these two curves (i.e., you cannot smoothly deform these two curves, without passing them through each other, in a way that changes L).

(a) Consider running unit "electric" currents along each polymer. Show using electrodynamics that the linking number, as defined above, is a topological quantity, and give a physical interpretation for what it counts.

Now let us study the connection with Chern-Simons gauge theories. Consider the partition function

$$Z(g) = \int \mathrm{D}\mathbf{x}_1(s)\mathrm{D}\mathbf{x}_2(s)\mathrm{D}A_1(\mathbf{x})\mathrm{D}A_2(\mathbf{x})\mathrm{e}^{-S_{\text{poly}}[\mathbf{x}_1,\mathbf{x}_2]-S_{\text{top}}[A_1,A_2,\mathbf{x}_1,\mathbf{x}_2]}$$

where g is a coupling constant, $\mathbf{x}_1(s)$ and $\mathbf{x}_2(s)$ are curves in space corresponding to the entangled polymers, A_1 and A_2 are 1-form fields in space, and $S_{\text{poly}}[\mathbf{x}_1, \mathbf{x}_2]$ is a "probability" term in the action which weights configuration curves $\mathbf{x}_1$ and $\mathbf{x}_2$ based on their interactions with each other. In general, this term may be quite complicated. However, the topological term is essentially a Chern-Simons term:

$$S_{\text{top}} = -\mathrm{i}g \int_{\mathbf{x}_1(s)} A_1 - \mathrm{i}g \int_{\mathbf{x}_2(s)} A_2 + \mathrm{i} \int_{\mathbb{R}^3} A_1 \wedge \mathrm{d}A_2.$$

(b) Explain why the propagators $\langle A_1 A_1 \rangle = \langle A_2 A_2 \rangle = 0$, and the propagator

$$\langle A_{1i}(p) A_{2j}(-p) \rangle = -\mathrm{i}\epsilon_{ijk} \frac{p_k}{|p|^2}.$$

Be sure to mention gauge invariance!

(c) Integrate out A_1 and A_2 and show that the entire contribution of the path integral is simply to break up the partition function into linking number sectors.

(d) How can we extract information about the probability that the linking number is L, given $Z(g)$?