Particle on Half Line

Famous examples of simple path integrals in quantum mechanics are for the free particle on $\mathbb{R}$ and S^1. Now, we discuss the path integral for particles on $\mathbb{R}^+$, the positive real axis.

We begin with the usual completeness identity:

$$\int_{\mathbb{R}^+} \mathrm{d}x |x\rangle\langle x| = 1,$$

and the identity

$$\langle x|x'\rangle = \delta(x-x').$$

Now, we use a spectral representation for $\delta(x-x')$ with functions that vanish at the boundary $x=0$:

$$\delta(x-x') = \frac{2}{\pi}\int_0^\infty \mathrm{d}k \sin(kx)\sin(kx').$$

(a) Show that we can extend x to $\mathbb{R}$ by using

$$\langle x|x'\rangle = \sum_{y=\pm x}\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi}\mathrm{e}^{\mathrm{i}k(y-x')+\mathrm{i}\pi(\Theta(-y)-\Theta(-x'))} = \delta(x-x') - \delta(x+x').$$

(b) By constructing the path integral just as we've done before, show that the propagator can be written as

$$\langle x_2, t_2|x_1, t_1\rangle = \int_{\substack{x(t_1)=x_1 \\ x(t_2)=x_2}} \mathrm{D}x \ \mathrm{e}^{\mathrm{i}S[x]+\mathrm{i}S_{\mathrm{boun}}[x]}$$

with $S[x]$ the classical action, and a "boundary" term

$$S_{\mathrm{boun}}[x] = -\pi\int \mathrm{d}t \ \delta(x(t))\dot{x}(t)$$

Here we treat x as existing on all of $\mathbb{R}$.

(c) Using the example of the free particle of mass m, show that the topological term in the path integral cancels over the paths which are not restricted just to $\mathbb{R}^+$, by explicitly constructing the propagator by performing the path integral.