Quantum Dots

Consider a quantum dot coupled to an external bath of charges. Roughly speaking, adding a charge to the quantum dot, or removing a charge, should cost an energy η. As we will see, regardless of the precise nature of the Hamiltonian of the quantum dot, we will see an effect called the Coulomb blockade, in which the density of states will develop a gap of 2η about 0. In the course of this problem, we will develop the physics of this strange effect.

Let's say that the density of states of the quantum dot in the absence of the external electrons is given by $\rho_0(\epsilon)$, and that it is given by $\rho(\epsilon)$ after we couple the system, and that the quantum dot has electron states characterized each by a Grassmann field ψ_α. We can determine the density of states in the usual way, by computing the Green's function $G_{\alpha\beta}(\omega)$, which is the Fourier transform of

$$G_{\alpha\beta}(\tau) = \frac{\displaystyle\int \mathrm{D}\overline{\psi}\mathrm{D}\psi \mathrm{e}^{-S[\overline{\psi},\psi]}\overline{\psi}_\beta(\tau)\psi_\alpha(0)}{\displaystyle\int \mathrm{D}\overline{\psi}\mathrm{D}\psi \mathrm{e}^{-S[\overline{\psi},\psi]}}.$$

The density of states can be found from the Green's function in the usual way.

While we don't know the exact eigenstates of the single electron Hamiltonian in the quantum dot, we'll find that the problem is in many ways very insensitive to this Hamiltonian. Let us assume we have a quantum dot at finite temperature T. A simple action we may write down is

$$S_{\text{dot}}[\bar{\psi},\psi] = \int\limits_0^\beta \mathrm{d}\tau \left[\sum_\alpha \bar{\psi}_\alpha\left(\frac{\partial \psi_\alpha}{\partial\tau} + (\epsilon_\alpha - \mu)\psi_\alpha\right) + \eta\left(\sum_\alpha \bar{\psi}_\alpha\psi_\alpha - N_0\right)^2\right].$$

Here N_0 is the expected number of electrons in the quantum dot, ϵ_α is the energy of state α in the quantum dot, and μ is the chemical potential of the electrons.

(a) Define a scalar (bosonic) auxiliary field $\chi(\tau)$ with the units of energy, and perform a HS transformation on the action so that it is quadratic in the fermionic fields.

(b) Explain why $G_{\alpha\beta}(\tau) = 0$ if $\alpha \neq \beta$.

(c) Express $\chi(\tau)$ in terms of Matsubara frequency components. Show that the gauge transformation

$$\psi_\alpha \to \psi_\alpha \mathrm{e}^{-\mathrm{i}\phi}$$
$$\bar{\psi}_\alpha \to \bar{\psi}_\alpha \mathrm{e}^{\mathrm{i}\phi}$$

where

$$\phi(\tau) \equiv \int\limits_0^\tau \left(\chi(\tau') - \chi_0\right)\mathrm{d}\tau'$$

makes the ψ part of the action only dependent on χ_0 (the zero-frequency component of χ), and make this gauge transformation in the path integral.

(d) Integrate out all frequency components of χ other than χ_0, the zero-frequency component, using the identity

$$\sum_{k=1}^{\infty} \frac{1-\cos(kx)}{k^2} = \frac{\pi|x|}{2} - \frac{x^2}{4} + \cdots .$$

Thus show that

$$G_{\alpha\beta}(\tau) \sim \mathrm{e}^{-\eta(\tau-\tau^2 T)} \int \mathrm{d}\chi_0 \mathrm{e}^{-\beta F(\mu - \mathrm{i}\chi_0)} G'_{\alpha\beta}(\tau;\chi_0) \mathrm{e}^{\mathrm{i}N_0\chi_0/T - \chi_0^2/4T\eta}$$

where $F(\mu)$ is an appropriately defined chemical potential-dependent free energy for the electrons and $G'_{\alpha\beta}$ is the Green's function of the electrons in the presence of the altered chemical potential $\mu - \mathrm{i}\chi_0$.

(e) Let us approximate that G' is independent of χ_0. We now proceed by doing a saddle point approximation for the χ_0 integral. Show that

$$\chi_0 = 2\mathrm{i}\eta \left[N_0 - \langle N \rangle_{\mu - \mathrm{i}\chi_0} \right].$$

Here, $\langle N \rangle_{\mu-\mathrm{i}\chi_0}$ is the expected electron number with the modified Hamiltonian, due to the χ_0.

(f) Suppose that $\eta \gg T$, and approximate Matsubara sums by integrals. Use the fact that

$$G'_{n\alpha} \approx \frac{1}{\mathrm{i}\omega_n + \epsilon_\alpha - \overline{\mu}}$$

where $\overline{\mu}$ is the renormalized chemical potential that can be found due to part (e), and Fourier transform the expression you found in part (d) to find an expression for $G_{n\alpha}$.

(g) Conclude from this expression that

$$\rho(\epsilon) = \rho_0\left(\epsilon - \eta \,\mathrm{sign}(\epsilon)\right) \Theta\left(|\epsilon| - \eta\right).$$

This is called the Coulomb blockade effect: note that there is an energy gap around the Fermi surface of width 2η where no electrons will be found!