Harmonic Oscillator in Magnetic Field

Consider the 2D harmonic oscillator

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\omega_0^2\left(x^2+y^2\right).$$

As usual we can define the annihilation operators a_x and a_y, and associated Hermitian conjugates. Furthermore, define

$$a_\pm \equiv \frac{a_x \pm \mathrm{i}a_y}{\sqrt{2}}.$$

(a) Show that $[a_\pm, a_\pm^\dagger] = 1$.

(b) Show that

$$H = \hbar\omega_0\left(a_+^\dagger a_+ + a_-^\dagger a_- + 1\right).$$

(c) Show that

$$L_z = xp_y - yp_x = \hbar\left(a_-^\dagger a_- - a_+^\dagger a_+\right).$$

We can now use this trick to use the harmonic oscillator creation/annihilation operators to find the spectra of Hamiltonians where the harmonic oscillator has an angular momentum term. Here's a simple example of this.

(d) Verify that the vector potential

$$\mathbf{A} = \frac{B\hat{\mathbf{z}} \times \mathbf{r}}{2}$$

corresponds to a constant magnetic field of magnitude B.

(e) Write down the Hamiltonian for the harmonic oscillator of charge q, placed in a magnetic field of constant strength B.

(f) By exploiting the tricks developed in parts (a)-(c), find the eigenvalues of this Hamiltonian. Express your answer in terms of $\hbar$, ω_0, and

$$\omega_{\mathrm{c}} \equiv \frac{qB}{2m}.$$