Jaynes-Cummings Hamiltonian

One of the chief problems of quantum computing is finding an effective way to measure the state of a quantum system without disturbing it too much. One proposal to do this is to use so-called cavity quantum electrodynamics. In this problem, we'll consider a simple model for such a system, which consists of a single atom trapped in a container, which can interact with photons at frequency ω. We may write the Hilbert space of the system to be given by $|n, s\rangle$ where $n = 0, 1, 2, \ldots$ counts the number of photons at frequency ω, and $s = \uparrow, \downarrow$ determines whether the atom is in spin up or spin down. The Hamiltonian for this system is called the Jaynes-Cummings Hamiltonian:

$$H = \frac{\hbar\omega}{2}\sigma_z + \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) + g\left(a^\dagger \sigma_- + a\sigma_+\right),$$

where $a^\dagger$ and a are the creation/annihilation operators for the photon, and the Pauli matrices act on the atomic state. Note that

$$\sigma_\pm = \frac{\sigma_x \pm \mathrm{i}\sigma_y}{2}.$$

(a) Define

$$N = a^\dagger a + \frac{\sigma_z}{2}.$$

Show that $[H, N] = 0$.

(b) Find the eigenvectors of H, and show that the eigenvalues are

$$E_{N,\pm} = \hbar\omega\left(N + \frac{1}{2}\right) \pm \sqrt{N+1}g.$$

For the remainder of the problem, you may assume that $g \gg \hbar\omega$, and ignore the ω terms in H.

Our goal is now to understand how this interaction can help us build a quantum computer. An ideal quantum computer would be able to perform the unitary transform

$$U = \begin{pmatrix} \cos(\alpha\theta) & -\mathrm{i}\sin(\alpha\theta) \\ \mathrm{i}\sin(\alpha\theta) & \cos(\alpha\theta) \end{pmatrix}$$

where α and θ are real constants. Our "cavity QED" quantum computer performs the unitary transform

$$\widehat{U} = \mathrm{e}^{-\mathrm{i}Ht/\hbar} \equiv \mathrm{e}^{-\mathrm{i}\theta H/g}$$

with $\theta \equiv gt/\hbar$. We'd like for U and $\widehat{U}$ to roughly correspond to each other.

To understand in what limits this will be true, we first use the fact that most quantum optics systems rely on *coherent states* for the photons, as opposed to eigenstates of H. Let $|\alpha\rangle$ be a coherent state for the photons, with $a|\alpha\rangle = \alpha|\alpha\rangle$. Assume that α is real, without loss of generality.

(c) Define the operator (on atomic states only) $\widehat{U}(n) = \langle n|\widehat{U}|\alpha\rangle$, and show that in the $|\uparrow\rangle, |\downarrow\rangle$ basis:

$$\widehat{U}(n) = \mathrm{e}^{-\alpha^2/2}\frac{\alpha^n}{\sqrt{n!}}\begin{pmatrix} \cos(\theta\sqrt{n}) & -\mathrm{i}\frac{\sqrt{n}}{\alpha}\sin(\theta\sqrt{n}) \\ \mathrm{i}\frac{\alpha}{\sqrt{n+1}}\sin(\theta\sqrt{n+1}) & \cos(\theta\sqrt{n+1}) \end{pmatrix}.$$

(d) As you may remember, the probability distribution for finding n photons in a coherent state is a Poisson distribution, which is sharply peaked for large α. Let $n = \alpha^2 + L\alpha$, and show that

$$\mathrm{e}^{-\alpha^2/2}\frac{\alpha^n}{\sqrt{n!}} \approx \frac{\mathrm{e}^{-L^2/4}}{(2\pi)^{1/4}}.$$

(e) For large α, we can approximate L as a continuous parameter. Verify

$$\int_{-\infty}^{\infty} \mathrm{d}L\, \widehat{U}(L)^\dagger \widehat{U}(L) = 1.$$

(f) Define the *fidelity* of this measuring device as

$$f = \min_{|\psi\rangle,\theta} \int_{-\infty}^{\infty} \mathrm{d}L\, \left|\langle\psi|U^\dagger \widehat{U}(L)|\psi\rangle\right|^2$$

where $|\psi\rangle$ refers to an atomic state. Taylor expand this quantity about $\alpha = \infty$, in powers of $1/\alpha$, to the lowest non-trivial order.

(g) Comment on the results of part (f) briefly; how would you want a "cavity QED" quantum computer to operate for optimal results? Can you make it as close to a perfect measuring device as you'd like?