Nuclear Magic Numbers

The energy required to extract a nucleon (proton or neutron) from the nucleus of some atom undergoes a dramatic decrease at the set of **magic numbers**:

$$2, 8, 20, 28, 50, 82, 126.$$

In this problem we will develop a simple model to explain this phenomenon. To begin with, let us consider modeling the nucleus as an isotropic simple harmonic oscillator in 3 dimensions:

$$H_0 = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{r}^2 - V_0.$$

V_0 is a constant offset which you can ignore for this entire problem, but it is large compared to $\hbar\omega$. The idea is that the harmonic oscillator is a simple model for the collective strong force holding the nucleus together. Consider filling this oscillator up with neutrons only (for simplicity, although protons would work just as well), which are spin 1/2 particles. The magic numbers then correspond to the number of neutrons at which one energy shell is completely full.

(a) What are the eigenvalues of H_0? What is the degeneracy of each state?

(b) What are the magic numbers for the harmonic oscillator?

As you should have found in part (b), this simple model does not work. The correct model will require adding an additional term to the Hamiltonian. Before we can do that, however, we have to understand angular momentum in the 3D harmonic oscillator. Since H_0 has rotational symmetry, it is clear that $[H, L^2] = 0$, and thus we can label eigenstates by both their energy E and their angular momentum l. The easiest way to understand the angular momentum of states in the harmonic oscillator is to consider the radial Schrödinger equation equation

$$E_{k,l}u_{k,l} = \left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{m\omega^2 r^2}{2} + \frac{l(l+1)\hbar^2}{2mr^2}\right)u_{k,l}$$

Here $u_{k,l} = rR_{k,l}$ is the radial part of the wave function at "level" k with angular momentum l. As with the hydrogen atom, we will want to study the behavior of the solutions of this differential equation to fix the energies, and most importantly their relationship with l.

(c) What are the asymptotics of u at large r? (For intermediate steps, drop the kl subscripts for conciseness). Extract this function, and then express u as $v(r) \times$ asymptotic term where $v(r)$ is a polynomial.

(d) Write

$$v(r) = r^s \sum_{q=0}^{\infty} a_q r^q$$

where $a_0 \neq 0$. Thus s is the power of the first non-zero coefficient in v. Find recursion relations between the elements a_q, using the radial Schrödinger equation.

(e) Explain why $s = l$.

(f) By showing that the wave function will be ill-defined if v does not have finite degree, determine the energies $E_{k,l}$. How many states at each degeneracy level of E have angular momentum l?

Now, we are ready to derive the magic numbers. The key observation is that the spin of the nucleon can couple to the overall angular momentum of the nucleon. This means we will set $H = H_0 + H_1$ where

$$H_1 = -\sum_{k,l} \lambda_{k,l} \frac{\mathbf{L} \cdot \mathbf{S}}{\hbar^2} \mathbb{P}_{k,l}.$$

Assume that $\lambda_{k,l}$ is a constant. Here $\mathbb{P}_{k,l}$ is a projection operator onto the subspace of eigenstates of H_0 with "index" k (from the previous few parts) and energy l.

(g) What are the eigenvalues of H_1?

(h) Show that appropriately chosen, but seemingly arbitrary, $\lambda_{k,l}$ can reproduce the magic numbers.

More precise models which allow for λ to be a function of r can reproduce the magic numbers with a less arbitrary choice of H_1, although there are usually smaller splittings leftover, thus rendering the magic numbers imperfect. Nonetheless, the basic idea you found in part (h) (of which eigenstates correspond to which "bands" in the magic numbers) is correct in the more complicated models.