Symmetry of the Hydrogen Atom

The massive degeneracy of the hydrogen atom is not an accident. In this problem, we will provide a simple algebraic argument which will lead to a deeper understanding of this degeneracy. In this problem, we will consider the Hamiltonian

$$H = \frac{p^2}{2m} - \frac{k}{r}.$$

Let's begin with an algebraic method. We need to begin by introducing something called the Laplace-Runge-Lenz vector. It is a classical constant of motion of the Kepler problem, and it is defined as

$$\mathbf{A} = \frac{\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}}{2m} - k\frac{\mathbf{r}}{r}.$$

This will also be the quantum mechanical expression for this object.

(a) Show that $\mathrm{d}\mathbf{A}/\mathrm{d}t = 0$, using the classical equations of motion and vector identities.

It is also a quantum conserved quantity, in that it commutes with the Hamiltonian: $[H, \mathbf{A}] = 0$. With a bit more work, one can derive:

$$[L_i, A_j] = \mathrm{i}\hbar\epsilon_{ijk}A_k,$$
$$[A_i, A_j] = -\mathrm{i}\hbar\epsilon_{ijk}L_k\frac{2H}{m}.$$

Since $\mathbf{A}$ and $\mathbf{L}$ commute with H, let us focus on a subspace of the hydrogen atom Hilbert space, in which $H = E < 0$. When restricted to this subspace, let us also define

$$\mathbf{B} \equiv \sqrt{\frac{m}{2(-E)}}\mathbf{A}.$$

(b) Show that if we define

$$\mathbf{M}_\pm \equiv \frac{\mathbf{L} \pm \mathbf{B}}{2},$$

that the components of $\mathbf{M}_+$ and $\mathbf{M}_-$ obey independent su(2) algebras.

(c) Derive the identity $\mathbf{L} \cdot \mathbf{A} = 0$, and conclude that $M_+^2 = M_-^2$.

With a bit of work, one can derive the identity

$$A^2 - k^2 = \frac{2H}{m}\left(L^2 + \hbar^2\right).$$

(d) Use this formula to show that, if $M_+^2 = M_-^2 = j(j+1)\hbar^2$, that

$$E = -\frac{mk^2}{2\hbar^2}\frac{1}{(2j+1)^2}.$$

Use the algebras we have developed to count the degeneracy of each of these energy states, and compare with the hydrogen atom.

In conclusion, we have shown that there is a hidden $\mathrm{SU}(2) \times \mathrm{SU}(2)$ symmetry of the hydrogen atom. This led us naturally to an understanding of the large degeneracy of the hydrogen atom.