1d Incommensurate Potential

Consider a Hamiltonian describing quantum particles hopping on a 1d lattice:

$$H = \sum_{n\in\mathbb{Z}} \left[-\frac{t}{2}\left(|n\rangle\langle n+1| + |n\rangle\langle n-1|\right) - a\cos\left(\phi + \frac{2\pi n}{\tau}\right)|n\rangle\langle n|\right].$$

where ϕ, τ, a and t are all positive constants. The t-term is a usual "hopping" term from site to site, and the a-term represents an onsite potential, whose period is τ. In this problem, we will be interested in the case where τ is an irrational number, so it is *incommensurate* with the underlying lattice.

We have previously seen that in such a model, if there is translation symmetry: if are allowed states at $E = 0$, the model is a toy model of a *metal*, and if not, the model describes an insulator. However, there is a second mechanism for obtaining an insulator: suppose that all energy eigenstates $|\psi\rangle$ are *localized*, in the sense that $|\psi\rangle$ has a well-defined norm. Then energy cannot be transported far across the sample in eigenstates, and it turns out this means that the model describes an insulator as well. Our goal in this problem is to show that the Hamiltonian above undergoes a transition from a metal to an insulator, when $t = a$.

To do this, consider some energy eigenstate $|\psi\rangle$, and define the quantities

$$\psi_n \equiv \langle n|\psi\rangle,$$
$$\phi_m \equiv \sum_{n\in\mathbb{Z}} \psi_n \mathrm{e}^{-\mathrm{i}(n+m)\phi - \mathrm{i}mn/\tau}.$$

(a) Show that there is a transformation which allows us to recover ψ_n from ϕ_m, and give the explicit formula.

(b) One can turn $H|\psi\rangle = E|\psi\rangle$ into an infinite-dimensional set of equations relating the ψ_ns:

$$E\psi_n = -\frac{t}{2}\left(\psi_{n-1} + \psi_{n+1}\right) - a\cos\left(\phi + \frac{n}{\tau}\right)\psi_n.$$

Show that there is an *almost equivalent* set of equations relating ϕ_ms to each other, but where the role of a and t has been reversed.[1]

(c) Argue that if the norm of $|\psi\rangle$ is finite, then the norm of $|\phi\rangle$ is infinite, and vice versa.

(d) Conclude that for $a < t$, the Hamiltonian above describes a metal, and if $a > t$, it describes an insulator.

It is a famous result that for particles on a 1 dimensional lattice, if the onsite energy (the a-term, in our problem), was described by a *random* function, no matter how weak a is compared to t, wave functions are always localized. Now, suppose that $\tau \ll 1$, and that $a < t$. We have just proven that the wave functions here are not localized – however, intuitively, the cosine function should begin to look quite "random". Apparently, however, the sequence of numbers generated by the cosine is not random. We can actually see why with a very simple argument. Consider the sequence of numbers

$$X_n = \cos\left(\phi + 2\pi\frac{n}{\tau}\right)$$

[1] I would do this by finding an expression for $E\phi_m$...

for $n = 1, 2, \ldots$. Define the sum

$$S_N \equiv \sum_{n=1}^{N} X_n.$$

(e) Show that $|S_N|$ is bounded by a finite number in the limit $N \to \infty$.[2]

(f) Conclude that the sequence X_n could not be generated by a random sequence, with probability one, as $N \to \infty$.

[2] Use the fact that $\cos\theta = \mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta})$.