quantum mechanics → symmetry ★★★★

Orthogonality Catastrophe

Let us consider a non-interacting system of N fermions, each of which is under the effect of a nondegenerate Hamiltonian H_0, whose eigenvalues are written as $E_1 < E_2 < \cdots$: i.e, the total Hamiltonian of the system is $H = H_{0,1} + \cdots + H_{0,N}$. Denote with $|0\rangle$ the properly normalized and antisymmetrized ground state of this system, and denote with $|i\rangle$ the single fermion eigenvector of H_0 corresponding to energy E_i.

Now, let us perturb the single fermion Hamiltonian to $H = H_0 + V$, for some small perturbation V, and denote $|0'\rangle$ as the ground state of this new Hamiltonian, again properly normalized and antisymmetrized. Define $|i'\rangle$ the single fermion eigenvector of H corresponding to energy E_i'.

The goal of this problem will be to compute

$$I \equiv \left|\langle 0|0'\rangle\right|,$$

i.e., the overlap between the two ground states. As we will see, for a typical system, when N is large, I is very small. This effect is called the **orthogonality catastrophe**.

(a) Find an exact expression for I, by expressing your answer in terms of the matrix $A_{ij} \equiv \langle i|j'\rangle$. The remainder of this problem deals with how to heuristically understand the behavior of I at large N.

(b) To proceed, it will help to prove the following mathematical theorem: let A be an $n \times n$ matrix, with column vectors e_i $(i = 1, \ldots, n)$ such that $\bar{e}_i \cdot e_i = 1$. Show that $|\det(A)| \le 1$.

(c) Combine the results of parts (a) and (b) to show that

$$I < \mathrm{e}^{-J}$$

where

$$J \equiv \frac{1}{2} \sum_{i \le N < j'} \left|\langle i|j'\rangle\right|^2 .$$

Suppose that we find $|i'\rangle$ by using first order perturbation theory. Suppose further that we have

$$\langle i|V|j\rangle \sim \gamma \mathrm{e}^{-\alpha|E_j - E_i|}$$

for typical states $|i\rangle$ and $|j\rangle$, and assume α is not "too small". This is a very crude approximation, but it will make the calculation simpler. Let $\rho(E)$ be the density of states for H_0 – assume that there are enough states, and N is large enough, that $\rho(E)$ may be approximated by a smooth function. Denote by E_F the Fermi energy: i.e., the energy E_N. Work only to lowest order in perturbation theory.

(d) Show that, so long as $\mathrm{d}\rho/\mathrm{d}E > 0$ "reasonably" fast, we find $J \to \infty$ as $N \to \infty$. This is the orthogonality catastrophe. Make any reasonable approximations.

This has important consequences in many condensed matter systems, particularly in experiments. We have shown that the ground state of a typical many-body condensed matter quantum system is incredibly sensitive to small perturbations. It turns out one of the practical consequences of this is that an experiment trying to measure a resistivity of a material may end up measuring the "contact resistance" at the leads where they are taking their measurement, due to the small perturbations of the leads!