Tunneling Out of a Well

In this problem, we will explore the dynamics of tunneling in a simple system. Let us begin by considering an arbitrary Hamiltonian H on an N-dimensional Hilbert space, with states labeled by $|n\rangle$, for $n = 1, \ldots, N$. This Hamiltonian has *translation symmetry*: i.e., if the generator of translations is given by

$$T = \sum_{n=1}^{N} |n\rangle\langle n+1|$$

(note $|N+1\rangle = |1\rangle$), then

$$[H, T] = 0.$$

H is completely arbitrary, at this point.

(a) What are the eigenvectors of H?

(b) Now, suppose we ask the following question: we prepare our quantum system to be in the state $|n = 1\rangle$. We pick a time uniformly at random from $0 \le t \le T$, where T is *very large* (take $T \to \infty$, if you like). Then, we measure the quantum system by observing what $|n\rangle$ we find. Assuming H has no degenerate eigenvalues, what is the probability that we find $|1\rangle$? In particular, if $|\psi(t)\rangle$ denotes the state of the system at time t, we want to compute

$$\mathrm{P}_T(1) = \frac{1}{T}\int_0^T \mathrm{d}t\, |\langle n = 1|\psi(t)\rangle|^2.$$

The answer to the previous part may fail for systems of interest. Assume that $N \gg 1$, and for simplicity let's now focus on the Hamiltonian

$$H = \sum_{n=1}^{N} \left[b|n\rangle\langle n| - a|n\rangle\langle n-1| - a|n\rangle\langle n+1| \right].$$

This simple Hamiltonian appears many times in condensed matter physics, as a simple model of electrons hopping on a crystal lattice.

(c) Verify that this Hamiltonian has degenerate eigenvalues, and find all of them.

(d) Repeat part (b), but now focusing on this Hamiltonian with degenerate eigenvalues. Use the approximation that $N \gg 1$ when appropriate. What changes? Is the new answer that you have found robust – what are the most generic changes that you can make to H while preserving the answer to this part?

(e) Assuming the Hamiltonian of the previous part, what is the time scale that we have to wait before we expect $\mathrm{P}_T(1)$ to approach $\mathrm{P}_\infty(1)$? Find the scaling of your answer with N, a and b (set $\hbar = 1$ if you like). This answers the question of the previous part in more detail, by telling us exactly what we meant by a very long time T.