PT Symmetric Quartic Hamiltonian

Consider the quantum Hamiltonian

$$H = \frac{p^2}{2m} - gx^4.$$

When restricted to the real line, this is certainly an ill-defined Hamiltonian. Surprisingly, if we choose to think of x and p as living in the complex plane, as opposed to the real line, it is possible to make sense of this H as having real, positive, and discrete eigenvalues. The deep reason for this is an unbroken PT-symmetry[1], but in this problem we will use a direct approach: we will show that the spectrum is real and discrete (the positive part would take more work) by mapping the Hamiltonian to another Hamiltonian with an identical spectrum.

(a) Let $\psi(x)$ be a stationary state with energy E. Write down the naïve eigenvalue equation for $\psi(x)$.

(b) As mentioned in the introduction, we can think of x as being a complex number, as opposed to just living on the real line. It turns out that we will ask that

$$\lim_{|x|\to\infty} \psi(x) = 0 \quad \left(0 > \arg(x) > -\frac{\pi}{3} \text{ or } -\frac{2\pi}{3} > \arg(x) > -\pi\right).$$

Consider the contour in the complex plane defined by

$$x = -2\mathrm{i}L\sqrt{1+\frac{\mathrm{i}y}{L}}.$$

with y a real parameter, and $L > 0$ a real constant with the units of length. Describe the contour $x(y)$. Verify that $\psi(y) \to 0$ as $|y| \to \infty$.

(c) Make the change of variables in the eigenvalue equation from x to y, and find the eigenvalue equation for $\psi(y)$.

(d) Now, perform a Fourier transform:

$$\phi(p) \equiv \int_{-\infty}^{\infty} \mathrm{d}y\ \mathrm{e}^{-\mathrm{i}qy/\hbar}\psi(y).$$

Write down the differential equation for $\phi(q)$, and verify that it is a second order equation, with a linear term with non-zero coefficient.

(e) Show that you can find a function $\alpha(p)$ for which

$$\phi(q) = \mathrm{e}^{\alpha(q)}\Phi(q),$$

and the eigenvalue equation for $\Phi(q)$ does not have a linear term.

[1] This means a symmetry under parity and time reversal. How this works exactly is not important for the purposes of this problem – we won't really use this fact in our solution.

(f) Show that an appropriate re-scaling of q, given by $z \equiv cq$ for some constant c, gives us the eigenvalue equation

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\Phi(z)}{\mathrm{d}z^2} + \left(4gz^4 - \hbar\sqrt{\frac{2g}{m}}z\right)\Phi(z) = E\Phi(z).$$

Explain why the eigenvalues of this should be real and discrete. (It also turns out they are positive – you don't need to argue this.)