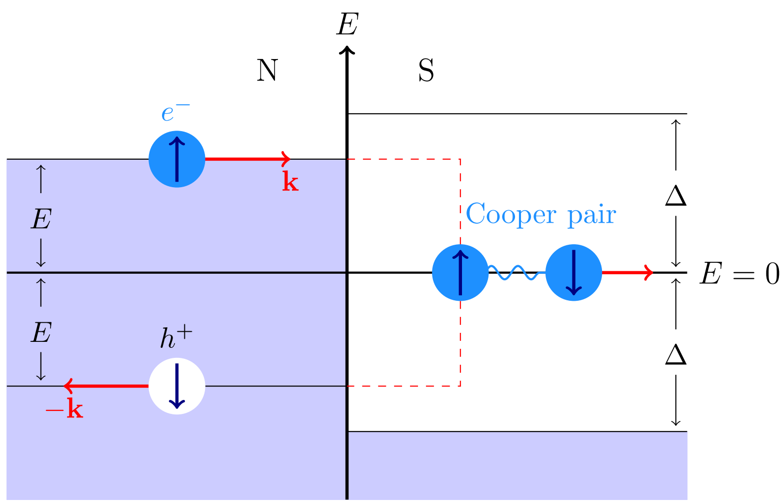
The BTK Model

Suppose we have an electron of mass m and charge $-e$, with energy E, incident upon a junction between a normal metal and a superconducting metal (called a S/N junction), both at temperature $T = 0$ (this makes statistical mechanics unnecessary for the problem). Electric current is not conducted normally inside of a superconductor: the charge carriers are *Cooper pairs* of electrons with opposing spins and momenta. The existing Cooper pairs inside of the superconductor exist at $E = -\Delta$. It takes an energy of 2Δ to break up a Cooper pair, at which point the electrons begin to propagate as *quasiparticles*, where they exist as both electrons and holes at the same time.

If an electron is incident with energy $E < \Delta$, then there is not enough energy to break up a Cooper pair. If there is no mechanism for reflection at the boundary, then the only possible outcome is for a hole to be reflected with energy $-E$, and for a Cooper pair to be formed inside of the superconductor, propagating with energy $E = 0$. This phenomenon is known as Andreev reflection, and is depicted below:



Ultimately, this is a simplistic picture. However, we can use a slightly modified version of Schrödinger's equation to look at this problem in a more formal light. This work was first done by Blonder, Tinkham and Klapwijk in 1982, and is named BTK theory after their work.

Let the wave function for our system be given by

$$\begin{pmatrix} \psi_{\text{electron}}(x,t) \\ \psi_{\text{hole}}(x,t) \end{pmatrix}.$$

Let the normal metal be located for $x < 0$, and the superconductor be located at $x > 0$. For reasons too complex to explain further here, the 2×2 Hamiltonian matrix for this system is given by

$$H = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \mu + Z\hbar\sqrt{\frac{2\mu}{m}}\delta(x)\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \Delta\Theta(x)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where μ is the chemical potential of electrons (an energy from statistical mechanics), Z is a parameter indicating the strength of the barrier at the junction[1] and $\Theta(x)$ is the step function:

$$\Theta(x) = \left\{ \begin{array}{ll} 1 & x \geq 0 \\ 0 & x < 0 \end{array} \right. .$$

For the entirety of the problem, assume we are looking for a time-independent solution (for all x) to this problem with energy $E > 0$, that has the following form:

$$\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{e}^{\mathrm{i}k_\mathrm{e}x}}_{\text{incident electron}} + \underbrace{a \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{e}^{\mathrm{i}k_\mathrm{h}x}}_{\text{reflected hole}} + \underbrace{b \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{e}^{-\mathrm{i}k_\mathrm{e}x}}_{\text{reflected electron}} \quad \text{for } x < 0,$$

$$\underbrace{c \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \mathrm{e}^{\mathrm{i}k_1 x} + d \begin{pmatrix} v_0 \\ u_0 \end{pmatrix} \mathrm{e}^{-\mathrm{i}k_2 x}}_{\text{quasiparticles}} \quad \text{for } x \geq 0.$$

(a) Find expressions for k_e, k_h, k_1, k_2, u_0 and v_0 in terms of E, μ, $\hbar$, Δ and m. Set u_0 and v_0 so that ${u_0}^2 + {v_0}^2 = 1$.

(b) Let $k_\mathrm{F} = \sqrt{2m\mu}/\hbar$. Approximate that $k_1 = k_2 = k_\mathrm{e} = k_\mathrm{h} = k_\mathrm{F}$ (this implies $\mu \gg E,\ \Delta$). By matching boundary conditions at $x = 0$, show that

$$\begin{aligned} a &= \frac{u_0 v_0}{\gamma}, \\ b &= \frac{({u_0}^2 - {v_0}^2)(Z^2 + \mathrm{i}Z)}{\gamma}, \\ c &= \frac{u_0(1 - \mathrm{i}Z)}{\gamma}, \\ d &= \frac{\mathrm{i}Z v_0}{\gamma}. \end{aligned}$$

where $\gamma = {u_0}^2 + ({u_0}^2 - {v_0}^2)Z^2$.

Suppose we accelerate the electrons using a voltage (as is common in the lab), so that $E = e\varphi$ for some voltage φ. It turns out that if we measure the current flowing across the junction, I, as a function of this voltage, we obtain:

$$\frac{\mathrm{d}I}{\mathrm{d}\varphi} = G = G_0(E)(1 + A(E) - B(E))|_{E = e\varphi}$$

where $A = |a|^2$ and $B = |b|^2$. G is called the *differential conductance*, and for low φ, $G_0(e\varphi)$ is essentially constant.

(c) Explain why the factor $1 + A - B$ appears in the expression for G.

(d) Plot G vs. $E = e\varphi$ for $0 \leq E \leq 3\Delta$, for both $Z = 0$ (no barrier) and $Z = 5$ (strong barrier). Normalize G so that $G \to 1$ as $E \to \infty$, in both plots. Comment on the results.

[1] In practice, there are two contributions to this: the first is simply due to the difference in Fermi velocities between the two metals, which are presumably different; the second is due to impurities, usually in the superconducting metal.

As Z increases, Andreev reflection becomes less likely (and harder to observe). Any spike in current that can be observed will mostly be the result of tunneling. But if Z is reasonably small, this effect can fairly easily be observed in a laboratory using a simple technique called *point-contact spectroscopy* in which a very small tip of non-superconducting metal is placed in proximity to a thin slab of superconducting metal, creating a junction for current to flow across. By measuring the current and the voltage drop across the junction, a plot of G vs. $e\varphi$ can be obtained and fit to the BTK theory to obtain fundamental numerical properties of the superconducting metal, such as the Fermi velocity, coherence length and pair potential Δ.