Bose-Einstein Condensates and Fluctuations

The Bose-Einstein condensation is a very famous phenomenon in physics, typically treated by using the grand canonical ensemble. However, the grand canonical ensemble turns out to be an inappropriate description for a full understanding of the condensate – this is due to the fact that there are remarkably few fluctuations in N_0, the number of particles in the condensate, as compared to what you naively expect. This issue was not resolved until 2013 by two mathematicians, and in this problem we will give a heuristic outline of their proof.

Let us begin by considering N non-interacting bosons at temperature T, placed in some quantum mechanical system. We are free to choose the ground state to have energy 0. Let us choose the excited states to have energies

$$0 < \epsilon_1 \leq \epsilon_2 \leq \epsilon_3 \cdots .$$

It turns out that many aspects of the Bose-Einstein condensate do not depend on precise details of the Hamiltonian, but only on the generic features of the density of states. Suppose that asymptotically, the density of states of the Hamiltonian is given by

$$\rho(\epsilon) \sim \lambda \epsilon^{\nu}$$

for $\nu > 0$ and $\lambda > 0$ some constants.

(a) The grand canonical ensemble is reasonable for predicting the critical temperature T_c at which a condensate forms. Give a heuristic explanation for this.

(b) Argue that T_c depends only on N, ν and λ and find an exact expression for it.

Unfortunately, we can't use the grand canonical ensemble for studying fluctuations of N_0, the number of particles in the condensate. The reason is of course that, what the grand canonical ensemble *actually* predicts is that the macroscopic N_0 number of particles simply leak back into the particle bath! So we'll have to stick with the canonical ensemble, and keep the total number of particles N fixed.

(c) Begin by assuming that $\nu > 1$. Using the fact that

$$\frac{\mathrm{P}(N_0 - 1, \ldots, n_i + 1, \cdots)}{\mathrm{P}(N_0, \ldots, n_i, \ldots)} = \mathrm{e}^{-\epsilon_i/T}$$

make a heuristic argument for the distribution underlying the random variables n_i, counting the number of particles in excited state i. Making the ansatz that it is reasonable to consider the n_i as independent, determine $\mathrm{Var}(N_0)$ (approximately) as a function of N and T/T_c. Comment on the fluctuations in the random variable N_0/N, and in particular give a physical motivation for the approximations made and the result.

(d) Explain why the argument you gave above breaks down for $\nu \leq 1$. By making the most simple possible fix to the problem, without detailed knowledge of the Hamiltonian, determine $\mathrm{Var}(N_0)$ as a function of N and T/T_c. What are the physical differences between this result and the case $\nu > 1$?

Somewhat remarkably, it turns out that the very crude arguments we used in the last two parts *essentially can be made rigorous*, up to $\mathrm{O}(1)$ factors.