Superradiance of Atoms

In this problem, we will explore a **superradiance** phase transition in a collection of N atoms, allowed to interact with light: a phase transition is marked by a dramatic increase in the light intensity emitted by atoms. Normally, if each atom is just emitting light incoherently (effectively, classically) then the light intensity is additive, and is proportional to N. In a superradiant phase, quantum mechanical coherence between the atoms and light means that the electromagnetic fields themselves are proportional to N – i.e., the light intensity is proportional to N^2. We will explore this phase transition through the use of quantum mechanics, and the canonical ensemble.

Let us consider a classic model, called the **Dicke model**. We consider a single harmonic oscillator of creation and annihiliation operators $a^\dagger, a$, which represents the number of photon excitations in a box: $|n\rangle$ represents a quantum state with n photons. This harmonic oscillator is coupled to $N \gg 1$ atoms, which we approximate as two level systems. If σ_i^I ($i = 1, 2, 3$, and $I = 1, \dots, N$) represents a Pauli matrix acting only the Hilbert space of atom I, then we write the Hamiltonian of the coupled atom-photon system as

$$H = \hbar\omega_{\mathrm{p}} a^\dagger a + \sum_{I=1}^{N} \left[\hbar\omega_{\mathrm{a}} \sigma_I^z + \frac{\lambda}{\sqrt{N}} \left(a^\dagger \sigma_I^- + a\sigma_I^+ \right) \right]$$

where the normalizing N factor will become clear by the end of the problem, and $\sigma_I^\pm = (\sigma_I^x \pm \mathrm{i}\sigma_I^y)/2$. ω_{p} represents the frequency of the photon, and ω_{a} represents the frequency of the transitions between the two atomic states we're considering.

Let us choose to measure energies in units where $\hbar\omega_{\mathrm{p}} = 1$. Alternatively, we denote $\omega_{\mathrm{a}}/\omega_{\mathrm{p}} \equiv \Omega$, $\lambda/\hbar\omega_{\mathrm{p}} \equiv \Lambda$, and $\beta = \hbar\omega_{\mathrm{p}} T^{-1}$, where T is the temperature, and express answers in terms of Ω, Λ, and β.

Our goal in this problem is to compute the free energy per particle,

$$f \equiv \frac{F}{N} = -\frac{\log Z(N,\beta)}{\beta} = -\frac{\log\ \mathrm{tr} \mathrm{e}^{-\beta H}}{\beta}.$$

To take the trace, it is convenient to use a coherent basis for the photons. This is a basis $|\alpha\rangle$, defined by a complex number α, so that $a|\alpha\rangle = \alpha|\alpha\rangle$, and it is complete:

$$\int \frac{\mathrm{d}^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = 1.$$

(a) Begin by defining $b = a/\sqrt{N}$ and $b^\dagger = a^\dagger/\sqrt{N}$. Argue that in the thermodynamic limit, $[b^\dagger, b] = 0$, and that we can replace $a^\dagger, a$ with $\overline{\alpha}, \alpha$, in the partition function, to leading thermodynamic order. What is the physical interpretation of this?

(b) Show that, approximately in the $N \to \infty$ limit:

$$Z(N,\beta) \sim \int_0^\infty \mathrm{d}y\ \mathrm{e}^{-N\beta y} \cosh\left(\beta\sqrt{\Omega^2 + \Lambda^2 y}\right)^N.$$

(c) Using the method of steepest descent, show that whenever $2\Omega > \Lambda^2$, the states of most weight have very few photons. What is the free energy per atom in this limit?

(d) Now, consider the case when $2\Omega < \Lambda^2$. Argue that in this case, there is a critical (inverse) temperature β_c, where for $\beta > \beta_c$, there is a transition to a superradiant phase. To argue that this phase is superradiant, explain why the intensity of light in the cavity will be proportional to N^2. Compute the free energy of this superradiant phase (you may not find an analytic expression).