Motion on SO(d) by Lagrange Multipliers

Previously, we have seen how to derive the equations of motion for the free d-dimensional rigid body by appealing to the group structure of the configuration space SO(d). There is a less mathematically elegant, but more direct approach, which employs Lagrange multipliers.

As before, let us consider the motion of a rigid body about the origin. If A describes the matrix in the configuration space SO(d), we have previously explained why the Lagrangian is

$$L = \mathrm{tr}\left[\frac{1}{2}\dot{A}K\dot{A}^{\mathrm{T}} - B\left(A^{\mathrm{T}}A - 1\right)\right]$$

where

$$K \equiv \int \mathrm{d}^d x \; xx^{\mathrm{T}}\rho(x).$$

Of course, the Lagrange multiplier B leads to a trivial term if we choose to express $\dot{A}$ in terms of the Lie algebra-valued velocities, but in this problem we will not do this. Instead, we will allow A to be completely arbitrary, and only impose the constraint that $A \in \mathrm{SO}(d)$ by the Lagrange multiplier (and continuity of the equation of motion).

(a) Explain why B is a symmetric matrix.

(b) Show that the equations of motion imply

$$A^{\mathrm{T}}\ddot{A}K - K\ddot{A}^{\mathrm{T}}A = 0.$$

(c) Define the matrix

$$\Omega \equiv A^{\mathrm{T}}\dot{A}.$$

Verify that Ω is antisymmetric.

(d) What is the equation of motion in terms of Ω?

(e) Verify by further manipulations that you can recover the equation we derived by group theoretic arguments earlier:

$$M^{ab}\dot{\omega}^b = f^{adc}M^{db}\omega^b\omega^c$$

where ω^b are the generalized angular velocities, M^{ab} is the generalized moment of inertia, and f^{adc} are the structure constants of SO(d).