## Flat Anisotropic Model

In this problem, we will look for very simple geometries which are solutions of Einstein's equation in the vacuum, in which the metric is only a function of a timelike coordinate t.

(a) Begin by showing that the metric may be put in the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \gamma_{ij}(t)\mathrm{d}x_i\mathrm{d}x_j.$$

by using diffeomorphism symmetry.

(b) Next, show that Einstein's equations imply<sup>1</sup>

$$\frac{1}{\sqrt{\gamma}}\partial_t \left(\sqrt{\gamma}\gamma^{ik}\dot{\gamma}_{kj}\right) = 0.$$
$$-\partial_t \left(\gamma^{ij}\dot{\gamma}_{ij}\right) = \frac{1}{2}\dot{\gamma}_{ij}\dot{\gamma}^{ij}$$

(c) Show that without loss of generality, we may choose the metric so that

$$ds^{2} = -dt^{2} + t^{2a}dx^{2} + t^{2b}dy^{2} + t^{2c}dz^{2},$$

with

$$a+b+c = a^2+b^2+c^2 = 1,$$
  
 $-\frac{1}{3} \le a \le 0, \quad 0 \le b \le \frac{2}{3}, \quad \frac{2}{3} \le c \le 1.$ 

This is called the **flat anisotropic model**, since we see that there must be spatial anisotropy.

Now, let us see what happens if we include matter into this model. Now generically, of course, this is not correct, but it may be approximately correct for small t, as we will show. One can argue that for small t, if we neglect the back reaction of matter on the geometry, the equations of fluid dynamics near the singularity are

$$\partial_t \left( \sqrt{-g} \epsilon^{3/4} u_t \right) = 0,$$
  
$$4\epsilon \partial_t u_i + u_i \partial_t \epsilon = 0.$$

Here  $u_{\mu}$  is the fluid 4-velocity, and  $\epsilon$  is the energy density of the fluid. We have also assumed that these solutions do not depend on time.

- (d) Determine how  $\epsilon$  and  $u_{\mu}$  scale with t, for small t.
- (e) How do the various components of  $T_{\mu\nu}$  scale with t at small t? By comparing to the geometric terms in Einstein's equation, justify that as  $t \to 0$  the geometric back reaction of this fluid is negligible.

<sup>&</sup>lt;sup>1</sup>Feel free to use the help of a numerical general relativity program! This may be a bit subtle.