The Focusing Theorem

In this problem, we will prove a very interesting result called the focusing theorem, which essentially says that a collection of light rays, or photons, will tend to converge together as they move through a curved spacetime. This theorem has been used by Hawking to prove theorems about black holes, and has many other interesting uses in general relativity.

First, we begin with geometric optics in curved spacetime. In geometric optics, we consider the vector potential to be

$$A^\mu(x) = a^\mu(x)\mathrm{e}^{\mathrm{i}S(x)},$$

and we define the wave vector

$$k^\mu \equiv \nabla^\mu S.$$

We assume that "$\nabla a \ll \nabla S$": i.e., to lowest order, we can only consider derivatives acting on S, when we compute ∇A. Finally, we define the intensity to be

$$I \equiv a^\mu \overline{a}_\mu$$

(note that a^μ is complex, in general) and up to some electromagnetic constants, it corresponds to the intensity of the electromagnetic wave.

Let's begin by proving 4 useful theorems for geometric optics.

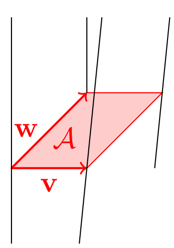
(a) Show that $a^\mu k_\mu \approx 0$.

(b) Show that $k^\mu k_\mu \approx 0$.

(c) Show that rays follow geodesics: i.e., $k^\nu \nabla_\nu k^\mu \approx 0$.

(d) Finally, show that $\nabla^\mu(k_\mu I) \approx 0$. Comment on the physical interpretation for this result.

Now, consider a small, rectangular bundle of rays propagating through space as shown below, and let $\mathcal{A}$ be the infinitesimal area of this bundle. We wish to follow the outlines of this bundle by propagating it along the rays that define it's edge, as shown in the picture. The edges are labeled by vectors $\mathbf{v}$ and $\mathbf{w}$, and these vectors satisfy $\mathbf{v}\cdot\mathbf{w} = \mathbf{v}\cdot\mathbf{k} = \mathbf{w}\cdot\mathbf{k} = 0$.



(e) Argue that $\mathcal{A}$ is a scalar under Lorentz transformations. Is it a scalar under diffeomorphisms?

(f) Working in a local Lorentz frame, show that $k^\mu \nabla_\mu \mathcal{A} = \mathcal{A}\nabla_\mu k^\mu$.

(g) Conclude that $k^\mu \nabla_\mu (I\mathcal{A}) = 0$. What physical principle does this correspond to?

(h) Let $\mathrm{d}/\mathrm{d}\lambda$ be the abstract vector representing ∇S: i.e., $\mathrm{d}/\mathrm{d}\lambda = k^\mu \nabla_\mu$. Show that[1]

$$\frac{\mathrm{d}^2\sqrt{\mathcal{A}}}{\mathrm{d}\lambda^2} = \left[\frac{1}{4}\left(\nabla_\mu k^\mu\right)^2 - \frac{1}{2}\nabla_\mu k_\nu \nabla^\mu k^\nu - \frac{1}{2}R^{\mu\nu}k_\mu k_\nu\right]\sqrt{\mathcal{A}}$$

(i) Working in a local Lorentz frame where $\mathbf{k} = \omega(\mathbf{e}_0 + \mathbf{e}_3)$, with $\mathbf{e}_3$ a space like direction and $\mathbf{e}_0$ a timeline direction, explain why

$$\frac{1}{4}\left(\nabla_\mu k^\mu\right)^2 \leq \frac{1}{2}\nabla_\mu k_\nu \nabla^\mu k^\nu.$$

(j) Consider a coordinate system in which $x^0 \equiv S$. Use the weak energy condition and the results of the previous parts and conclude that

$$\frac{\mathrm{d}^2\sqrt{\mathcal{A}}}{\mathrm{d}\lambda^2} \leq 0.$$

This result is called the focusing theorem. Describe physically what it means.

[1] You will want to use the identity for $[\nabla^\mu, \nabla^\nu]V^\rho$, for a generic vector field ρ.