Warp Drive

The warp drive was first described as a fictional method for traveling faster than light in science fiction. Amazingly, it is actually not so hard to write down a metric for a spacetime which describes a warp drive. In this problem, we will do this task, and describe what the setbacks of this proposal for faster-than-light travel are.

The key insight into understanding the warp drive is as follows: while nothing can *locally* travel faster than light, one can imagine a special region R of the spacetime in which a geodesic would travel faster than light would in the region outside of R With this thought in mind, consider the metric

$$\mathrm{d}s^2 = -\left(1 - v_\mathrm{s}^2 f(r_\mathrm{s})^2\right)\mathrm{d}t^2 - 2v_\mathrm{s} f(r_\mathrm{s})\mathrm{d}x\mathrm{d}t + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

where

$$r_\mathrm{s} \equiv \sqrt{(x - x_\mathrm{s}(t))^2 + y^2 + z^2},$$

$x_\mathrm{s}(t)$ is some pre-specified function with velocity

$$v_\mathrm{s} = \frac{\mathrm{d}x_\mathrm{s}}{\mathrm{d}t}$$

and f is the function

$$f(x) = \frac{\tanh(\sigma(x+R)) - \tanh(\sigma(x-R))}{2\tanh(\sigma R)}.$$

The precise form of f is not particularly important for this problem, and we will just use this one for "simplicity".

(a) Show that the metric describes a spacetime which is essentially flat, except within a "bubble" of radius R (in the spatial directions, at any fixed time t). This is where the warp drive is.

(b) Write down the geodesic equation for a particle in this spacetime. Show that for a massive particle, there is a geodesic corresponding to the trajectory $x(t) = x_\mathrm{s}(t)$. This will correspond to a particle sitting inside the warp drive.

(c) Is there any time dilation along the geodesic $x = x_\mathrm{s}(t)$?

Now the key thing to notice is that $x_\mathrm{s}(t)$ is a property of spacetime itself, and is therefore completely unconstrained! Therefore, we can always choose $v_\mathrm{s} \gg 1$. Since the speed of light is 1 far outside of the bubble, the particle in the warp drive is thus moving "faster than light", although not locally.

As you might expect, such solutions of the equations of general relativity are "bad." One reason they might be bad is that an observer may see negative energy density in a region of space: to many people, this seems "unphysical" (although with quantum effects, arguably not impossible).

(d) Let u^μ be the 4-velocity of the particle inside of the warp drive. Show that the energy density this particle observes everywhere in spacetime is negative: i.e.,

$$T_{\mu\nu}u^\mu u^\nu \leq 0.$$

Thus in order to create a usable warp drive, one would need to figure out how to create negative energy density in spacetime, which is of course a very difficult task.