# Oxygen Isotopes and Evaporating Water

Sometimes, chemistry can slightly change depending on the isotope of a given element present. As an example, we consider the oxygen molecules in water molecules, $H_2O$. There are 2 isotopes of oxygen of interest, $^{16}O$ and $^{18}O$. Let us denote $r = N_{18}/N_{16}$ to be the ratio of the number of water molecules with $^{18}O$ to the number of water molecules with $^{16}O$. This ratio turns out to depend slightly on the phase in equilibrium: the liquid phase contains slightly more heavy oxygen than the vapor phase, with the difference described by a small parameter $\epsilon$:

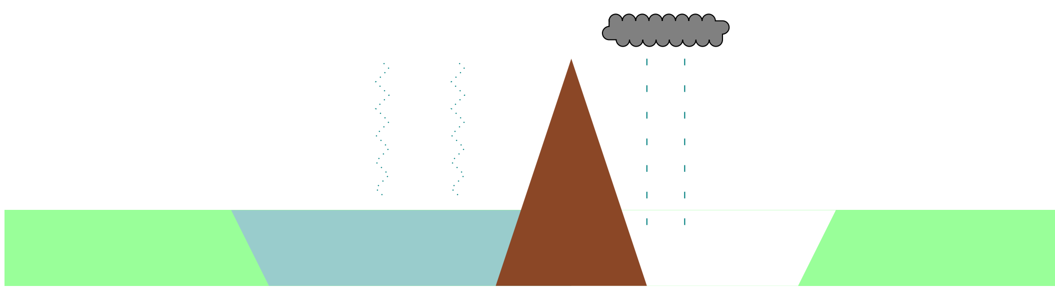
$$r_{\mathrm{l}} = (1+\epsilon) r_{\mathrm{v}}$$

Now, suppose we have a very large lake in which the water vapor at the surface of the lake is in equilibrium with the water in the lake, which has isotope ratio $r_0$. However, water vapor rises, as it is a gas. As it rises, water condenses out of the vapor and drifts back to the lake with a rate of $\alpha$ per unit length: if the density of water vapor is $n_{\mathrm{v}}$:

$$\frac{\mathrm{d}n_{\mathrm{v}}}{\mathrm{d}z} = -\alpha n_{\mathrm{v}}.$$

(a) Assuming that as the liquid water condenses out of the water vapor, it is always in equilibrium, find an expression for $r_{\mathrm{v}}(z)$.

Now, let us consider the following situation. As the water vapor rises, the wind blows it to the east. A tall mountain of height $h$ sits in the way – assume that any of the water vapor which reaches a height $h$ immediately blows over the mountain, where it condenses into water and falls as rain into a now empty lake basin.



(b) As the eastern lake just starts to fill, what is $r$? To get a sense for how serious these isotope effects could be, let's use some realistic numbers: $r_0 \approx 2\times10^{-3}$, $\epsilon \approx 10^{-2}$, $\alpha \approx 5\times10^{-4}\ \mathrm{m}^{-1}$. What happens for $h = 500$ m, or $h = 1500$ m?

(c) Suppose that the volume of the western lake was originally $V_0$, and that it is now $V$. Find expressions for $r_{\mathrm{W}}(V)$ and $r_{\mathrm{E}}(V)$, the isotope ratios in the western and eastern lakes, respectively. Sketch your results. You do not need to plug in values for the variables.

Now, suppose that the "lake basin" on the eastern side of the mountain is high enough that the moisture in fact falls as snow, and lake 2 is in fact a glacier, made up of solid ice.

(d) Let $t_0$ represent the thickness of the glacier after all the water from lake 1 has been deposited as snow onto the glacier. After this happens, a scientist extracts an ice core of thickness $t_0$, which allows her to determine the value of $r$ as a function of the distance $t$ from the (newly deposited) surface of the ice. Find an expression for $r(t)$.

Effects such as the one described in this problem are of extreme importance to climate scientists, as the parameters described in this problem are sensitive to temperature (an effect which we will ignore for this problem). If we assumed, for example, that the lake was refilling with water with a constant $r_0$, then variations in $r(t)$ give us clues as to the temperature of the water-vapor equilibrium at various points in time.