Lévy Distributions

The central limit theorem is a "universal" result of probability theory, but there are some distributions where the central limit theorem fails. In particular, consider a large sum of random variables with a scale free distribution of exponent $1 < \gamma < 3$. We cannot employ the central limit theorem because the variance of the scale free distribution is infinite, and therefore we must employ another technique.

Let us consider the Lévy distribution, which is defined according to

$$\mathrm{p}(x) \sim \int_{-\infty}^{\infty} \mathrm{d}k \, \mathrm{e}^{\mathrm{i}kx - a|k|^\mu}$$

for $a > 0$ and $0 < \mu < 2$. Note that the Lévy distribution is essentially defined by its characteristic function.

(a) Consider a random variable X distributed according to the Lévy distribution as above. Estimate the PDF of the distribution for large values of X. To do this, you can approximate that for large X, the integral is dominated by the region of small momenta, and justify expanding $\mathrm{e}^{-a|k|^\mu}$ to lowest nontrivial order. You should find

$$\mathrm{p}(X) \sim \frac{1}{X^{1+\mu}}.$$

(b) Now, let us consider the average of a sum of a large number of Xs drawn iid from the Lévy distribution:

$$Z = \frac{1}{N} \sum_{i=1}^{N} X_i.$$

Determine the asymptotic distribution of Z, and verify that

$$\mathrm{p}(Z) \sim \frac{1}{Z^{1+\mu}}$$

for Z large. Also, discuss the form of the coefficient of proportionality, and comment on its implications.

(c) What happens for $\mu = 2$? Argue that only when $0 < \mu < 2$ do we find power law distributions.[1]

We have argued heuristically that a sum of random variables with infinite variance results in a distribution which maintains its heavy tailed power law distribution. This can be made far more precise but it is a very mathematically intense task, and the key intuition is given in this problem.

Note also that this has important consequences for many systems of practical interest. Recall that wealth is typically distributed as a scale free random variable with $\gamma \approx 2$, and thus has infinite variance. This implies that the average wealth of a group of people is extremely large and is dominated by only a few individuals.

[1] In fact when $\mu > 2$ the Lévy distribution is not a well defined probability distribution – the Fourier integral is negative at some values of x.