Lévy Flights

Consider a list of N iid random variables $X_1, \ldots, X_N$, such that

$$\mathrm{P}(X_i > x) = \max(1, x)^{-\nu}.$$

In this entire problem, assume that $N \gg 1$. When $0 < \nu < 1$, the resulting behavior of

$$S \equiv \sum_{i=1}^{N} X_i$$

dramatically changes. This phenomenon is heuristically called a Lévy flight.

(a) Show that for $\nu \leq 1$, $\langle S \rangle = \infty$. This is why the phenomenon is "exotic".

(b) For what values of ν is $\mathrm{Var}(S) = \infty$? When will the (Gaussian) central limit theorem hold?

Just because $\langle S \rangle$ is formally infinite does not mean we have no way of understanding the behavior of the sum. The trick is just that we have to view things from a different perspective. Let's define $J_1, \ldots, J_N$ such that J_1 is the largest of the X_i, J_2 is the second largest, etc. Then define

$$K_i \equiv \frac{J_i}{S},$$
$$Z_i \equiv \frac{J_i}{J_1}.$$

(c) Determine the distribution on J_1, making reasonable approximations for large N. What is the typical scale of J_1?

(d) It should be clear that although the distribution on J_i is "badly behaved", the distributions on K_i and Z_i are certainly well behaved. Although we'd ultimately like to understand the distribution on K_i, it is substantially easier to find the distribution on Z_i. Find the probability density function of Z_i for $i \ll N$, making any reasonable approximations when N is large.

(e) Prove that for $\nu < 1$, $\mathrm{P}(K_1 > 0) = 1$. Comment on the implications of this result for S, and compare to the usual central limit theorem.

(f) The origin of the term of Lévy *flight* comes from the use of these random variables to model random walks of migratory animals, or animals foraging for food. If this is indeed a reasonable model for foragers, what does it imply about their search behavior?

(g) Comment on the $\mathrm{P}(K_1 = 0)$ (after taking the limit $N \to \infty$), in the case $\nu = 1$. You do not need to be particularly formal but make a convincing argument.