Random Matrices

Let $|\alpha\rangle$ represent the state of a nucleus, and H be the Hamiltonian matrix for the nucleus. We do not necessarily know that these are eigenstates of H, although since H is a real-valued symmetric matrix we know it has real eigenvalues. In reality there are a huge number of states for a complex nucleus like ^{238}U, far more than we can ever hope to measure exactly. In the 1950s, Wigner and Dyson realized that we could model a system like this using a matrix with a large number of random entries. That way, even if we don't get any particular transition energy right, we may expect on average to see the overall behavior of the system; e.g., within a keV or so, how much resonance do we expect to see? Thus, the basic question we want to answer is the following: given a random H, what is the probability distribution of the eigenvalues?

Let A be a real-valued symmetric $n \times n$ matrix with i.i.d. $\mathcal{N}(0,1)$ entries; not surprisingly, A is called a **random matrix**. Assume that n is very large, but finite.

(a) Show that (for a generic symmetric matrix B) and $k \in \mathbb{N}$,

$$\mathrm{tr}(B^k) = \sum_{i=1}^{n} {\lambda_i}^k.$$

(b) Let λ be an eigenvalue of the random matrix A. Show that

$$\langle \lambda^k \rangle = \begin{cases} n^{k/2} \cdot \dfrac{1}{1+k/2} \begin{pmatrix} k \\ k/2 \end{pmatrix} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}.$$

This is a challenging combinatorics problem, so be careful!

Suppose we could find a probability distribution on λ that had the properties found in (b); then we might say that that is indeed the same as the distribution of the eigenvalues of A themselves. Mathematically, that is not technically correct and further proof would be required, but for this problem that is all that is necessary. Consider the following probability distribution on λ:

$$\mathrm{p}(\lambda) = \frac{1}{\pi\sqrt{n}} \sqrt{1 - \frac{\lambda^2}{4n}}\,\Theta(2\sqrt{n} - |\lambda|).$$

(c) Show that this probability distribution has the same moments as you found in part (b).

The fact that this is the proper distribution to describe the eigenvalues of A is known as **Wigner's semicircle rule** (as the graph of $\mathrm{p}(\lambda)$ is a semicircle). While we didn't prove it, this is also true for many different types of random variables A_{ij}.

Random matrix theory is a fairly modern field of mathematics with applications to number theory and computational linear algebra, as well as nuclear physics.