The Holtsmark Distribution

Suppose that the distribution of stars in space is isotropic, such that the number density of stars is the universe is n at all points in space. Approximate that all stars have the same PDF $\mathrm{p}(M)$ for their mass M, and can be treated as point masses. Let $\mathbf{r}_i$ be the position of a star, as measured by an observer sitting at the origin. The gravitational field felt by the observer is given by the formula

$$\mathbf{g} = \sum_i GM_i \frac{\hat{\mathbf{r}}_i}{{r_i}^2}.$$

(a) Show that the characteristic function of $\mathbf{g}$ is given by[1]

$$\begin{aligned}\tilde{\mathrm{p}}_{\mathbf{g}}(\mathbf{k}) &= \exp\left[-\int_0^\infty \mathrm{d}^3\mathbf{r}\mathrm{d}M \int n\mathrm{p}(M)\left(1 - \mathrm{e}^{-\mathrm{i}GM\mathbf{k}\cdot\hat{\mathbf{r}}/r^2}\right)\right] \\ &= \exp\left[-\frac{4n}{15}\left(2\pi GM_0 k\right)^{3/2}\right]\end{aligned}$$

where ${M_0}^{3/2} \equiv \langle M^{3/2}\rangle$.

(b) Show that

$$\mathrm{p}(\mathbf{g}) = \frac{1}{2\pi^2 g^3} F\left(\frac{g}{(4n/15)^{2/3} 2\pi GM_0}\right),$$

where the function $F(x)$ is defined by

$$F(x) = \int_0^\infty \mathrm{d}z\; z\sin z \mathrm{e}^{-(z/x)^{3/2}}.$$

$\mathrm{p}(\mathbf{g})$ is referred to as the *Holtsmark distribution.*

(c) Plot $F(x)$, and use it to give a fairly rough numerical estimate for the most probable result if g is measured, assuming that $G = 6.7\times10^{-11}$ J · m/kg, $M_0 = 2.0\times10^{30}$ kg and $n = 10^{-50}$ stars/m^3.

[1] First consider a fixed volume V of space with a fixed number N of stars, with $N = nV$. Then take the limit that V becomes all of space. Use the identity $\int_0^\infty \mathrm{d}z\,(z-\sin z)z^{-7/2} = \frac{4\sqrt{2\pi}}{15}$ to conclude.