The Greenhouse Effect

The **greenhouse effect** refers to the phenomenon that the Earth's effective temperature can be raised by "trapping" electromagnetic thermal radiation. The trapping is caused by the interactions of low energy photons with gases in the atmosphere – in particular, it is believed, H_2O and CO_2. In this problem, we will derive a simple theoretical model explaining the greenhouse effect in mathematical terms.

We will first begin by looking at a seemingly unrelated problem, that of "gambler's ruin". This is a classic Markov chain from the theory of probability, but we will only need to derive a simple result here. The basic problem is the following: suppose that I have a random walk on the lattice $\{0, 1, \dots, N\}$. If my random walk starts at $X_0 = k$, and at each time step I have an equal probability of taking a step left or right, what is the probability I will hit $X = 0$ before I hit $X = n$?

(a) We solve this problem as follows. Define ρ_k to be the probability that one reaches $X = N$ before $X = 0$, given that one starts at $X_0 = k$. Find a set of recursive relations between the ρ_k.

(b) Solve these equations and show that

$$\rho_k = \frac{k}{N}.$$

Now, we return to the greenhouse effect. We can approximate the dynamics of a single photon by a vertical random walk through the atmosphere. We will show later on that the density of the atmosphere is well approximated by

$$n(z) = n_0 \mathrm{e}^{-z/a},$$

with $z = 0$ corresponding to ground level, n_0 to the density near the surface, and a to some characteristic length scale. Now, we basically need to figure out in what chunks we should break up the atmosphere so that the photon has an equal probability of moving up or down by the time it is leaving the block. To do this, we need information about how readily the photon interacts with gases, and this is conveniently described by the cross section σ_0. Using basic scattering theory, we can argue that the appropriate block size for the atmosphere should be on the order of

$$\Delta z = \frac{1}{\sigma n(z)}$$

(c) Using this equation, show that even if the atmosphere has infinite height, the probability q of a photon escaping the atmosphere is finite, and is given by

$$q = \frac{1}{a\sigma_0 n_0}.$$

(d) Find q for a famous greenhouse gas, CO_2, using $a \approx 8000$ m, $\sigma_0 = 3.7 \times 10^{-23}$ m^2, and $n_0 = 10^{22}$molecules/m^3.

Now, we are ready to model the greenhouse effect. Let us call $I_0 \approx 1361$ W/m^2 the intensity of light bathing the Earth (due to the Sun), and q_0 the initial escape probability for a photon reflected off of the Earth's surface. We call $\epsilon \approx 0.7$ the "absorptivity" of the Earth (the fraction of incident radiation which does not reflect off of the atmosphere). Label with σ the Stefan-Boltzmann constant. Trying crudely to compare this model with realistic data, we would choose $q_0 \approx 0.6$.

(e) Using this data, find the effective average surface temperature of the Earth, T_0. Does it seem accurate?

(f) Now suppose that the effective concentration n_0 of gas increases to n'. Realistically, you could think of this as doubling only the concentration of CO_2, e.g., which was the gas we used when making the above model. In terms of variables, find the new equilibrium temperature T' of the Earth.

(g) If $n' = 2n_0$, find T'/T_0. What is T' as an absolute temperature?

As you might expect, this model dramatically exaggerates the real greenhouse effect. The dominant flaw in this model is that only a few photons will have the proper frequencies to interact strongly with molecules in the atmosphere. A more realistic model would try and take this into account, among other things (for example, considering feedback processes in density and interaction strength with temperature, etc.) but this is far outside the scope of this problem.