Growing Surfaces

How does snowfall cover terrain of varying height? How does sand or dirt poured over some surface settle? These are very simplistic questions, but the answers, for a realistic surface, are a bit subtle. In this question, we will explore the behavior of a surface growing in such a manner.

A simple model of a growing surface is as follows: let $h(x,t)$ be the height of the surface at position x (on some d dimensional terrain) at time t. Then

$$\frac{\partial h}{\partial t} = u_0 + \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2$$

where u_0, ν and λ are positive constants. u_0 is the (presumed constant, in space and time) rate of deposition of new material on the surface. ν represents the tendency of the materials on the surface to diffuse around (reducing the height), and λ represents the fact that particles will tend, after landing, to "roll" down the surface, simply due to gravity.

By setting $h \to h - u_0 t$, we can remove the constant term in the equation. (From now on, assume we have done this). The **Cole-Hopf transformation**

$$z = \mathrm{e}^{\lambda h/2\nu}$$

helps make the nonlinear PDE manageable.

(a) Show that the Cole-Hopf transformation reduces the nonlinear PDE for h to the diffusion equation.

(b) If the initial height of the surface (e.g., as the snow begins to fall) is $h(x,0) = h_0(x)$, what is the height of the surface at time t?

(c) Often it is reasonable to take the limit $\nu \to 0$ (this corresponds to the fact that the particles tend to stay in the same place after they have settled). In this limit, show that

$$h(x,t) \approx \sup_{x'} \left[h_0(x') - \frac{(x-x')^2}{2\lambda t} \right].$$

(d) Describe a simple "graphical" method for sketching the solutions to the above equation. Show how a surface grows for large t, assuming some arbitrary, complicated $h_0(x)$.

For a general surface, $h_0(x)$ is going to be some extremely complicated function. For many examples in nature, $h_0(x)$ will look, roughly speaking, random. One crude way to characterize the randomness is by an exponent χ:

$$\left\langle (h_0(x+z) - h_0(x))^2 \right\rangle \sim |z|^{2\chi}.$$

where $|z| \gg a$. For mountains, empirically we find $\chi \approx 0.7$, e.g.

(e) This randomness has interesting implications. Show that "information" about the surface spreads as

$$z \sim t^{1/(2-\chi)},$$

where we mean that the surface "knows" about the local maximum of h_0, within distances given by z above, at time t. Compare this to the case of simple diffusion.