Random Walks with Shrinking Steps

In this problem, we will consider some cute features of random walks with shrinking steps. In particular, pick a $0 < \lambda < 1$, and then define

$$X = \sum_{n=1}^{\infty} \lambda^n Z_n$$

for Z_n, $n = 1, 2, \ldots$ iid random variables with $\mathrm{P}(Z_n = \pm 1) = 1/2$. If we had picked $\lambda = 1$ this would be the simple random walk in one dimension. For $\lambda < 1$, the random walk has shrinking steps.

Denote with $\mathrm{p}_\infty(x)$ the PDF of X as $n \to \infty$. You may assume that this limit exists for the purposes of this problem.

(a) Show that $\mathrm{p}_\infty(x)$ has compact support for any $0 < \lambda < 1$.

(b) Show that if $\lambda < 1/2$, $\mathrm{p}_\infty(x)$ has support on a Cantor set.[1]

(c) Now, take $\lambda = 1/2$. By computing the Fourier transform of the PDF of X after n steps, and then taking the limit $n \to \infty$, show through clever cancellations that you find a *uniform* distribution on X at infinite time.

[1] Think about the following question: how far can you get from X_n at time step $n + 1$?