The Virial Theorem and Dark Matter

Often times, very simple and elegant theorems in statistical mechanics can have incredibly profound consequences. Here is a lovely example from classical statistical mechanics:

(a) Prove that Z is independent of the choice of coordinates: i.e., if the transformation $q_i \to q_i'$, $p_i \to p_i'$ is canonical,

$$Z[H(q_i', p_i')] = Z[H(q_i, p_i)].$$

This provides a very elegant proof of the **virial theorem**, a theorem in classical mechanics, for a system in thermal equilibrium. Let the Hamiltonian of a system of interacting non-relativistic particles be

$$H = \sum_i \frac{\mathbf{p}_i{}^2}{2m_i} + V(\mathbf{r}_i).$$

The virial theorem states that

$$\left\langle \sum_i \frac{\mathbf{p}_i{}^2}{m_i} \right\rangle = \left\langle \sum_i \frac{\partial V}{\partial \mathbf{r}_i} \cdot \mathbf{r}_i \right\rangle.$$

Classically, the averages were time averages; for us, they are phase space averages in the canonical ensemble. We can call upon an ergodic theorem to justify interchanging the two types of averages, as is usually done.

(b) Show that the transformation $q_i' = \lambda q_i$, $p_i' = p_i/\lambda$ is canonical.

(c) Combine parts (a) and (b) to prove the virial theorem.

Now, let's turn to a totally different area of physics. In 1933, Fritz Zwicky was attempting to calculate the mass of the Coma cluster, a galaxy cluster about 10^{24} m away from Earth. He computed this mass in two ways: first, by simply guessing from the size of the galaxy, and second by using the virial theorem. You might find the following facts helpful: when looking at this cluster in a telescope, the observed angular distance from one end to the other is about 10^{-4} radians; the average density of a galaxy is about 10^{-20} kg/m^3; the average velocity of a galaxy is 10^6 m/s; Newton's gravitational constant $G \approx 10^{-10}$ J · m/kg^2. You need only worry about orders of magnitude for the remainder of the problem.

(d) Estimate the mass of the Coma cluster from its size; then use the virial theorem. Compare your answers; by how many orders of magnitude do they differ?

(e) Why would the existence of dark matter resolve the discrepancy of part (d)? (Say more than just "because now I can make the two masses equal.")

Modern estimates put the percentage of the universe made up of dark matter at about 80-90%. We still have very little idea as to what makes up dark matter.