

Electricity in the Nervous System

Biological systems frequently exploit electrochemistry to perform work or aid in other useful tasks. As evidence of this, many cells are observed to maintain fairly significant potentials across their membranes – we'll quantify exactly what we mean by significant in this problem. The cell transports ions such as Na^+ or K^+ through specialized ion-channel proteins across the membrane in order to do this.

One of the most spectacular displays of electricity in biology is the electromagnetism of the nervous system. For a large animal to rapidly (within milliseconds) send signals across its body, it cannot rely on chemical/diffusive processes alone: it must use electricity.

Let's begin by assuming that we have charged particles, each with charge q . The voltage inside the cell is constant everywhere and is V_2 , and the concentration of charged particles is c_2 ; outside, the voltage/concentration is V_1/c_1 respectively. Assume that the charged particles are free to travel across the membrane. For this problem, we work with a cell at temperature $T \approx 300$ K, and always assume thermal equilibrium.

- (a) Starting from reasonable assumptions, derive the **Nernst equation**:

$$V_2 - V_1 = -\frac{k_B T}{q} \log \frac{c_2}{c_1}.$$

- (b) The Nernst equation gives us a voltage scale for biological electricity. By using the electron charge $q = 1.6 \times 10^{-19}$ C, determine the voltage scale for biological electricity. Recall that $k_B \approx 1.4 \times 10^{-23}$ J/K.
- (c) It is not uncommon to see the voltage drop across the cell membrane (called the **action potential**) to surpass 70-80 mV in cells on millisecond time scales. Compare to the voltage scale above; is this impressive?

In reality, charges cannot flow through the membrane easily, and they must pass through the ion-channel proteins. Let's model the ion-channel protein as a 2 state system, with an open state and a closed state. The energy difference between the two states is given by, for a single protein,

$$E_{\text{open}} - E_{\text{closed}} = \epsilon - QV$$

where $V = V_2 - V_1$ is the voltage drop across the membrane, and Q is an effective charge of the protein. The reason the energy difference is dependent on Q is that the ion-channel proteins themselves have slight electric dipoles which can orient themselves favorably and lower their energy when strong voltage drops (and thus, electric fields) are present.

- (d) Show that the fraction of ion-channels that are open as a function of V , $f(V)$, is given by the following Fermi function like expression:

$$f(V) = \frac{1}{1 + e^{Q(V_m - V)/k_B T}}.$$

and find V_m .

- (e) Typically, $Q \approx 12q$, where q is the electron charge. Approximately how sensitive is the ion-channel; i.e., roughly speaking, if you want to flip the gate from open to closed, by how much do you need to change the voltage?

As stated in the introduction, the most dramatic and beautiful demonstration of biological electricity is given by the nervous system in animals. Here, electrical signals propagate down the **axon** – a specialized part of the neuron cell. We can model the axon as a long, thin cable: indeed, axons can get to be as long as 1 m in larger animals. Let us assume the interior of the axon has resistivity ρ , the axon has a cross-sectional area of A , and that the membrane has capacitance per unit length c . We now allow the voltage drop V across the membrane to be a function both of time t and position x along the axon.

- (f) Assuming that all ion-channel proteins are closed, show from conservation of electric charge that the voltage $V(x, t)$ obeys a diffusion PDE:

$$\frac{\partial V}{\partial t} = D \frac{\partial^2 V}{\partial x^2}$$

and find D .

- (g) Modify the diffusion equation above to include the fact that the ion-channel proteins can allow charges to flow through, assuming that the number of ions (all of charge q) which flow through an protein per unit time is given by γV (for some unknown constant γ), and that there are n_c ion-channel proteins per length. Allow for the fact that not all proteins are open/closed!
- (h) Solving the nonlinear diffusion equation above is impossible to do by hand. But one can get a very good sense for the rough behavior of it by simple arguments. One can get very far, in fact, purely from dimensional analysis. Using dimensional analysis only, determine up to a dimensionless constant the speed at which signals propagate along the axon.