Hemoglobin

Often times proteins work in a cooperative fashion. For example, a protein may only function properly (fold into the proper shape) if a pair of molecules are simultaneously attached to the protein. We model this in statistical physics by assuming that the energy is not simply a linear sum of terms. One of the classical examples of a protein which behaves cooperatively is hemoglobin, the protein responsible for carrying O_2 in the bloodstream. A hemoglobin molecule can bind up to 4 O_2s at once, and ideally a hemoglobin molecule has either 4 or 0 oxygens (either it transports oxygen towards the cells, or is heading back to the lungs to be replenished).

We will work with a slightly simpler model for hemoglobin, which consists of a protein with 4 possible configurations, which we can label by (σ_1, σ_2), with $\sigma_i = 0$ $(i = 1, 2)$ representing the state where site i on the protein is unoccupied (0) or occupied (1) by oxygen. Suppose that the energy for this configuration is

$$E = (\epsilon - \mu)(\sigma_1 + \sigma_2) + J\sigma_1\sigma_2.$$

ϵ corresponds to the energy per site of attaching an oxygen, and J is the "interaction energy", and the chemical potential

$$\mu = \frac{1}{\beta} \log \frac{c}{c_0}$$

where c is the O_2 concentration, and c_0 is a reference value.

(a) Find the partition function Z.

(b) If the system is at temperature T (with $\beta = 1/k_B T$), determine $\langle n \rangle$, if $n = \sigma_1 + \sigma_2$.

(c) Find the expected value of $f_2 = \sigma_1\sigma_2$, the fraction of proteins with both sites occupied.

(d) Determine the limiting behavior of $\langle n \rangle$ and $\langle f_2 \rangle$, both when c is very small, and when c is very large.

(e) Plot $\langle n \rangle$ and $\langle f_2 \rangle$ on the same plot, as a function of c for 4 different sets of values of ϵ and J: one for each possible combination of ϵ positive/negative, and J positive/negative. Comment on the results.

(f) As suggested above, a biological system would likely prefer to either have this protein either bound to 0 or 2 oxygens. How should we pick ϵ and J so that this is most likely?