Intensity Dependent Refractive Index

In nonlinear optics, one of the most classic phenomena is an intensity dependent refractive index, where

$$n = n_0 + n_2 I$$

where I is the intensity of the electromagnetic field. The goal of this problem is to use statistical physics to explain this result.

Consider the molecule carbon disulfide (CS_2), which is a rod-shaped molecule with a nontrivial polarizability: per molecule, the electric dipole moment is given by

$$\mathbf{p} = \alpha_\parallel \mathbf{E}_\parallel + \alpha_\perp \mathbf{E}_\perp .$$

where $\mathbf{E}_\parallel$ is the component of the electric field parallel to the molecule, and $\mathbf{E}_\perp$ is the perpendicular component. It is a result, approximately, from optics, that

$$n^2 \approx 1 + N\langle\alpha\rangle$$

where N is the number of molecules per unit volume, and $\langle\alpha\rangle$ is defined by

$$\langle\alpha\rangle = \left\langle \frac{\mathbf{E}\cdot\mathbf{p}}{\mathbf{E}\cdot\mathbf{E}} \right\rangle$$

Assume that these molecules are at temperature T, with T "large".

(a) Find the energy of the molecule, assuming that its axis makes an angle θ with a constant electric field $\mathbf{E}$.

(b) Compute $\langle\alpha\rangle$, assuming T is large, using statistical physics.

(c) Conclude that

$$n_0 = \sqrt{1 + \frac{N(\alpha_\parallel + 2\alpha_\perp)}{3}},$$
$$n_2 \approx \frac{N(\alpha_\parallel - \alpha_\perp)^2}{45 {n_0}^3 \epsilon_0 c k_{\mathrm{B}} T}.$$