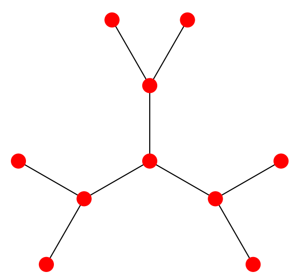
Ising Model on a Tree

In this problem, we will explore the Ising model on a tree, which has very curious behavior. A tree is a graph with no loops, and can be constructed as follows. Start with a root node, and attach "leaves" – nodes with only one edge, which connects to the root. Then, repeat the process, attaching new leaves to the old leaves. However long one continues this process, we see the key property of a tree, which is that *there is a unique path* to get between any two nodes. This makes many models in statistical physics exactly solvable.

For all of this problem (except part (c)), let us assume that the tree is a so-called k-regular tree, where every node in the tree has k edges. So to build this tree, we start with a node, add k leaves to it, and then recursively add $k-1$ leaves to each of our current leaves. If we repeat this process M times before stopping, we will say that the tree has a total level of M; the number of edges between any given node and the root node is referred to as the level of that node. An example of such a regular tree with $k=3$ and $M=2$ is shown below.



As usual, the Hamiltonian for the Ising model is

$$H = -\sum_{i\sim j} s_i s_j.$$

Here a site i refers to a node in the tree, $s_i = \pm 1$ is an Ising spin on that site, and $i \sim j$ means that there is an edge between nodes i and j. We have assumed for simplicity that the coupling is set to 1, as it simply sets the appropriate scale for the inverse temperature β.

(a) Let Z_M be the partition function for a tree with M levels. Show that

$$Z_{M+1} = Z_M[2\cosh\beta]^{N_{\mathrm{leaf}}}$$

where N_{leaf} is the number of leaves added to get from the tree with M levels, to the tree with $M+1$ levels.

(b) Find an exact expression for the partition function Z, and the free energy per site $\mathcal{F} \equiv F/N$. Show that $\mathcal{F}(\beta)$ is an always-differentiable function; should there be any phase transitions in this model?

(c) Would your answer to the previous part change for a more general tree structure?

Now, suppose that we alter the boundary conditions, so that all of the leaves of the tree are fixed to have spin up.

(d) Explain why we can exactly evaluate the partition function similarly to in parts (a) and (b). However, there is a slight complication. Let H_i be the Hamiltonian which only includes interactions between two spins in level $\leq i$; let $Z_{(i)}(s_0, \ldots, s_{i+1})$ be the partition function, only summing over the spins in levels $> i$. Explain why

$$Z_{(i)} = \mathrm{e}^{-\beta H_i} \prod_{i+1} \sum_{s_{i+1}=\pm 1} \mathrm{e}^{\beta s_i s_{i+1} + \beta h_{i+1} s_{i+1} + c_{i+1}}$$

and describe how one finds c_{i+1} and h_{i+1}. In the sums above, s_j represents a generic spin at level j; the product over $i+1$ corresponds to a product over each spin at level $i+1$. Conclude by describing how to find the partition function Z of the full tree.

(e) Show that[1]

$$\tanh \frac{\beta h_{i-1}}{k-1} = \tanh(\beta h_i) \tanh(\beta).$$

(f) Show that there is a phase transition to an ordered phase, and find the transition point β_c.[2]

(g) Explain the discrepancy between the answers to parts (b) and (f).

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[1] You may find this identity helpful: $\dfrac{\cosh(A+B)}{\cosh(A-B)} = \exp\left[2\operatorname{arctanh}\left(\tanh(A)\tanh(B)\right)\right]$.

[2] Remember to discuss stability of any phase.