Breaking Bones

In this problem, we will explore the physics which occurs in bone fracture. One "typical" way to imagine breaking a bone is to free fall from some height h through Earth's gravitational field g, and then impact a hard surface. Assume that the hard surface is very thick, and that it will not fracture or in any way behave outside of the elastic regime. We'll want to get a sense of how large a height h is required to break a bone, and why.

Let us collect the relevant material properties for this problem. In the set-up of our problem, the bone you will break is a leg bone, which has a length of about $L \approx 0.5$ m, and an effective cross-sectional area of $a \approx 0.02$ m^2. Bone will fracture when it is subject to stress greater than $\sigma_c = 0.15$ GPa. Suppose that you fall on a hard "rock" surface, with Young's modulus $E \approx 50$ GPa, and speed of sound $c \approx 1000$ m/s. Finally, suppose that your mass is $m = 60$ kg, and $g = 10$ m/s^2, and that the area over which the impact force is spread (your feet) is about 0.1 m^2. Do not plug in for these values, until instructed, at the end of the problem.

(a) Suppose you jump from a height h and are in free fall. When you hit the ground, how fast are you going?

(b) Let z be the distance that you fall into the rock, and τ be the time it takes for you to come to rest. Relate z and τ, and then show that the force on your body during impact is (if you approximate it to be constant during deceleration)

$$F = mg\frac{h}{z}.$$

(c) During the impact time, we can estimate that the rock behaves as a slab of thickness $c\tau$ – this is because the "shock" of your impact can only propagate through the material at the speed of sound. Estimate z by requiring that F be a consequence of Hooke's Law for the elastic solid.[1]

(d) Conclude by finding a formula for the stress in the bone during impact, and then a formula for the maximal height h_c at which you can fall without breaking the bone.[2]

(e) Find a formula for the critical height h_c. At this height, what are z and τ? Comment on your answer – are you surprised by the result?

(f) One mechanism that animals use to increase h_c is to bend their knees during impact. For a typical fall of $h \sim 1$ m, this will increase z to be about 1 cm. What is h_c now? Compare to the answer from the previous part.

[1] Note this is technically inconsistent, since we required that the force be constant during the fall in part (b). However, accounting for these changes will only lead to an O(1) correction factor, which we neglect for simplicity.

[2] Be sure to account for the fact that the bone has a different cross-sectional area than the feet which make contact with the surface!