continuum mechanics \rightarrow elastic solids

Wrinkling

Consider a thin sheet of thickness h, made of a solid with Young's modulus E and Poisson's ratio ν . In this problem, we want to consider how such a sheet can wrinkle and tear, using primarily scaling arguments. Below we see the geometry of a wrinkling solid, with thickness in the z-direction h, length in the x-direction L, and width in the y-direction W. We take $h \ll W \ll L$.



Let us begin by discussing the wrinkling of the thin sheet depicted above. Suppose we stretch the solid by a uniform strain $s_{xx} = s_0$. We observe that wrinkles begin to appear in the y direction, as shown in the figure. If we can approximate that the boundary of the region at constant y are fixed, and use the thin plate approximation that the state of the sheet is determined by $\zeta(x, y)$, with ζ the z-displacement, then we can determine the state of the solid by finding the minimizers of the functional

$$F[\zeta(x,y)] = \int \left[\frac{\tau}{2} \left(\frac{\partial \zeta}{\partial x}\right)^2 + \frac{B}{2} \left(\frac{\partial^2 \zeta}{\partial y^2}\right)^2 + b \left(\frac{1}{2} \left(\frac{\partial \zeta}{\partial y}\right)^2 - a\right)\right] \mathrm{d}x \mathrm{d}y$$

Let's begin by explaining why the statement above is true.

- (a) Explain the form of the first two terms in F, by determining B and τ , up to constant factors, in terms of the parameters s_0 , E, ν , h, L and W. Think about the energy required to bend or stretch an elastic sheet.
- (b) The final term in F is a Lagrange multiplier term, with Lagrange multiplier b. Explain what the constraint represents, and why it takes the form it does, and find the value of a in terms of the previous set of parameters.
- (c) Find the equations of motion. Plugging in for the ansatz

$$\zeta(x, y) = \zeta_0 \sin\left(\phi_x + k_x x\right) \sin\left(\phi_y + k_y y\right),$$

determine self consistent equations for the parameters.

- (d) Find how k_y depends on the remaining parameters, including k_x .
- (e) How much do we need to stretch the sheet before we expect wrinkling to first occur?