## Wrinkling

Consider a thin sheet of thickness $h$, made of a solid with Young's modulus $E$ and Poisson's ratio $\nu$. In this problem, we want to consider how such a sheet can wrinkle and tear, using primarily scaling arguments. Below we see the geometry of a wrinkling solid, with thickness in the $z$-direction $h$, length in the $x$-direction $L$, and width in the $y$-direction $W$. We take $h \ll W \ll L$.


Let us begin by discussing the wrinkling of the thin sheet depicted above. Suppose we stretch the solid by a uniform strain $s_{x x}=s_{0}$. We observe that wrinkles begin to appear in the $y$ direction, as shown in the figure. If we can approximate that the boundary of the region at constant $y$ are fixed, and use the thin plate approximation that the state of the sheet is determined by $\zeta(x, y)$, with $\zeta$ the $z$-displacement, then we can determine the state of the solid by finding the minimizers of the functional

$$
F[\zeta(x, y)]=\int\left[\frac{\tau}{2}\left(\frac{\partial \zeta}{\partial x}\right)^{2}+\frac{B}{2}\left(\frac{\partial^{2} \zeta}{\partial y^{2}}\right)^{2}+b\left(\frac{1}{2}\left(\frac{\partial \zeta}{\partial y}\right)^{2}-a\right)\right] \mathrm{d} x \mathrm{~d} y
$$

Let's begin by explaining why the statement above is true.
(a) Explain the form of the first two terms in $F$, by determining $B$ and $\tau$, up to constant factors, in terms of the parameters $s_{0}, E, \nu, h, L$ and $W$. Think about the energy required to bend or stretch an elastic sheet.
(b) The final term in $F$ is a Lagrange multiplier term, with Lagrange multiplier $b$. Explain what the constraint represents, and why it takes the form it does, and find the value of $a$ in terms of the previous set of parameters.
(c) Find the equations of motion. Plugging in for the ansatz

$$
\zeta(x, y)=\zeta_{0} \sin \left(\phi_{x}+k_{x} x\right) \sin \left(\phi_{y}+k_{y} y\right),
$$

determine self consistent equations for the parameters.
(d) Find how $k_{y}$ depends on the remaining parameters, including $k_{x}$.
(e) How much do we need to stretch the sheet before we expect wrinkling to first occur?

