Oscillator-Electron Interaction

This problem will guide you through a *rare* example of an exactly solvable interacting model in quantum physics, which concerns the interaction of a single electron with a single bosonic oscillator.[1] The Hamiltonian of this model is

$$H = \omega a^\dagger a + \left(\eta + \gamma\left(a + a^\dagger\right)\right) c^\dagger c,$$

with $[a^\dagger, a] = 1$ describing a bosonic oscillator, and $\{c^\dagger, c\} = 1$ describing the electron, which is a fermionic excitation. The goal of this problem will be to compute the Green's function for the electron:

$$G(t) \equiv -\mathrm{i}\langle 0|c(t)c^\dagger(0)|0\rangle$$

with $|0\rangle$ the vacuum state with no fermionic/bosonic excitations.

(a) Let's begin by proving some identities. Suppose that, for some special Hermitian operator R, we define

$$\overline{A} \equiv \mathrm{e}^{\mathrm{i}R} A \mathrm{e}^{-\mathrm{i}R}.$$

Show that $\overline{AB} = \overline{A}\,\overline{B}$.

(b) Show that

$$\overline{A} = A + \mathrm{i}[R, A] - \frac{1}{2}[R, [R, A]] + \cdots = \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!} \underbrace{[R, \ldots, [R, A]]}_{n \text{ times}}.$$

(c) Now, find an R such that[2]

$$\overline{H} = \omega a^\dagger a + \left(\eta - \frac{\gamma^2}{\omega}\right) c^\dagger c.$$

(d) Find $\overline{a}$, $\overline{a^\dagger}$, $\overline{c}$ and $\overline{c^\dagger}$.

(e) Use the "R transformation" to show that

$$G(t) = -\mathrm{i}\exp\left[-\mathrm{i}\left(\eta - \frac{\gamma^2}{\omega}\right)t - \frac{\gamma^2}{\omega^2}\left(1 - \mathrm{e}^{-\mathrm{i}\omega t}\right)\right].$$

(f) Show that this Green's function implies that the electron will occupy a spectrum of equally spaced energy states, and that the probability of finding the electron in these states is approximately given by an exponential distribution of width γ^2/ω.

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[1] The model can be extended to include any number of bosonic oscillators in a fairly straightforward fashion. For simplicity, we will work with one oscillator, which captures the essentials of the method.

[2] Try to "complete the square".