probability theory $\rightarrow$ stochastic differential equations

## Volatility in Financial Markets

This problem will develop a simple model for a stochastic differential equations model of a stock market which will suggest the existence of very large fluctuations in stock prices. Consider a stock whose price as a function of time is $S(t)$, and let $V(t)$ be a function which describes the variance of fluctuations in time. We will assume that $S$ and $V$ obey the Itō stochastic differential equation

$$
\frac{\mathrm{d}}{\mathrm{~d} t}\binom{S}{V}=-\binom{0}{\gamma(V-\bar{V})}+\sqrt{V}\left(\begin{array}{cc}
1 & 0 \\
\alpha \rho & \alpha \sqrt{1-\rho^{2}}
\end{array}\right)\binom{\dot{W}_{1}}{\dot{W}_{2}}
$$

with $W_{1}$ and $W_{2}$ independent Brownian motions. Note that the time in $\sqrt{V(t)}$ lies just before the time in the Brownian motions. The physical intuition for this equation is simple: $\gamma$ represents the rate at which volatility decays to its natural rate $\bar{V}$, while $\alpha$ determines the relative strength of volatility fluctuations, and $\rho$ determines the extent to which fluctuations in volatility are coupled to fluctuations in stock prices.
(a) Let $X(t)=\log S(t)$. Find the SDE for $X$ and $V$.
(b) Find the Fokker-Planck equation for the SDE found in part (a).
(c) It turns out that we must have

$$
\alpha^{2} \leq 2 \gamma \bar{V}
$$

for this model to be well-defined: i.e., for $V(t) \geq 0$. Show that this is indeed the case by considering the behavior of solutions to the Fokker-Planck equation near $V=0$.
(d) Ignoring fluctuations in $X$, show that the equilibrium distribution for $V$ is given by a $\Gamma$ distribution:

$$
\mathrm{p}_{\mathrm{eq}}(V)=\frac{\mu^{\nu}}{\Gamma(\nu)} V^{\nu-1} \mathrm{e}^{-\mu V}
$$

Find expressions for $\mu$ and $\nu$ in terms of $\bar{V}, \rho, \alpha$ and $\gamma$. Comment on the solution - is the volatility predictable?
(e) Now, let us turn to the full Fokker-Planck equation. By Fourier transforming both $X$ and $V$, show that the exact solution is given by ${ }^{1}$

$$
\begin{aligned}
\mathrm{p}\left(t, X, V \mid 0,0, V_{0}\right)= & \int_{-\infty}^{\infty} \frac{\mathrm{d} k}{2 \pi} \mathrm{e}^{\mathrm{i} k X} \int_{-\infty}^{\infty} \frac{\mathrm{d} q}{2 \pi} \mathrm{e}^{\mathrm{i} q V} \\
& \times \exp \left[-\frac{V_{0}}{\alpha^{2}}\left(B \frac{C \mathrm{e}^{B t}+1}{C \mathrm{e}^{B t}-1}-A\right)+\frac{\gamma \bar{V}(A-B) t}{\alpha^{2}}-\frac{2 \gamma \bar{V}}{\alpha^{2}} \log \frac{C-\mathrm{e}^{-B t}}{C-1}\right]
\end{aligned}
$$

[^0]with
\[

$$
\begin{aligned}
& A=\gamma+\mathrm{i} \alpha \rho k \\
& B=\sqrt{A^{2}+k(k-\mathrm{i}) \alpha^{2}} \\
& C=1-\frac{2 \mathrm{i} B}{\alpha^{2} q-\mathrm{i}(A-B)}
\end{aligned}
$$
\]

(f) Averaging over $V$, find the limiting form of $\mathrm{p}(t, X)$ when $t$ is large, up to a $t$ dependent normalization constant. ${ }^{2}$
(g) Show when $x$ is very small, that the distribution is approximately a Gaussian distribution. What are the mean and variance?
(h) Show when $x$ is very large that the distribution is exponential, not Gaussian. What is the scale of $x$ required to see a crossover from Gaussian to exponential behavior?
(i) What is the distribution of stock price $S=\mathrm{e}^{x}$, both in the regime where $S \approx 1$ ( $x$ very small) and $S$ very large?

We have thus seen that fluctuating volatility will lead to heavy tails in stock prices. Interestingly, real stock market data suggests that the "universal function" (up to the various constants) found in (f) for the probability distribution of stock prices is quantitatively accurate.
${ }^{2}$ The following identity may be useful: $\int_{0}^{\infty} \mathrm{d} k \cos (a k) \mathrm{e}^{-b \sqrt{k^{2}+1}} \sim \frac{\mathrm{~K}_{1}\left(\sqrt{a^{2}+b^{2}}\right)}{\sqrt{a^{2}+b^{2}}}$.


[^0]:    ${ }^{1}$ There are a lot of steps required to do this and it can be a bit tricky, so be very thorough. It's not a big deal if you can't reach this final answer. You may find helpful that the solution to the differential equation $\dot{x}=a x^{2}+b$ with $a$ and $b$ constant coefficients is $x(t)=-\frac{1}{a} \frac{\mathrm{~d}}{\mathrm{~d} x} \log (\cos (\sqrt{a b} x)+c \sin (\sqrt{a b} x))$ with $c$ a constant of integration.

