Consumption over a Lifetime

In the theory of economics, consumers maximize their **utility** (or happiness, satisfaction, etc.) over their lifetime.[1] Assume that a person lives for N years, and is working for all of them, and that their income is Y_n in year n. Also, assume that at the beginning of year n, they have money A_n at their disposal from before, and that they spend C_n in money during year n: this is their consumption during year n. The goal then is to maximize

$$\text{total utility} = \sum_{n=1}^{N} u(C_n)$$

where $u(C)$ is an instantaneous utility function, which has the property that (for all reasonable values of C, at least):

$$\frac{\mathrm{d}u}{\mathrm{d}C} > 0, \qquad \frac{\mathrm{d}^2u}{\mathrm{d}C^2} < 0.$$

What makes this problem a bit subtle is that Y_n (and, therefore, C_n) are random variables.[2]

(a) Explain why the consumer will maximize their expected total utility by choosing, in year k:

$$\frac{\mathrm{d}u(C_k)}{\mathrm{d}C} = \left\langle \frac{\mathrm{d}u(C_n)}{\mathrm{d}C} \middle| k \right\rangle, \text{ for all } n \geq k.$$

Here the conditional probability is taken with respect to the knowledge the consumer has in year k. In this way, we see that the derivative of instantaneous utility (often called the **marginal utility** in economics) is a martingale.

The simplest non-trivial choice for $u(C)$ is

$$u(C) = C - \frac{\alpha}{2}C^2$$

for a constant $\alpha > 0$.

(b) Show that, in this special case,

$$C_k = \frac{1}{N-k}\left(A_k + \sum_{n=k}^{N} \langle Y_n \mid k \rangle\right)$$

(c) Also show that

$$C_k - C_{k-1} = \frac{1}{N-k}\sum_{n=k}^{N}\left(\langle Y_n \mid k\rangle - \langle Y_n \mid k-1\rangle\right)$$

[1] Arguably, people focus on increasing instantaneous utility, but we neglect that assumption in this problem.

[2] It would be nice, although possibly less interesting, if you knew your future income!

(d) Conclude that C_k (i.e., consumption itself) is a martingale.[3] Discuss the consequences of this behavior: do consumers in this model "care" about uncertainty?

(e) What do you think are the consequences of this on consumer behavior. If you give a temporary tax break, for example, do you think it would have an immediate effect on the economy?

The simple quadratic model of utility does help to understand a lot of facts in macroeconomics: e.g., the point on tax breaks made above. But other empirical tests suggest that this model breaks down.

If the quadratic model is too simple, then the physicist's way to proceed would be to consider

$$u(C) = C - \frac{\alpha}{2}C^2 + \frac{\beta}{3}C^3.$$

(f) Describe how the cubic term will alter consumer behavior. Do not assume that β is positive. Do you predict **precautionary saving** against future uncertainty?

[3] In economic jargon, this often is referred to as the certainty-equivalence behavior, or the fact that consumption is a "random walk." This is not technically true of course, because the deviations from expected behavior do not need to be i.i.d. random variables, but jargon has a bad habit of sticking around...