Johnson Noise

If you don't actively power a circuit with a power source, are there still voltages that appear? The answer is yes – this was an experimentally observed fact in 1928; it is named **Johnson noise** after its discoverer. This voltage comes from the random motion of the electrons inside the circuit. Let's consider the simple case of a resistive circuit element with resistance R, as well as a non-zero inductance L. We can write the equation of motion as

$$L\mathrm{d}I(t) + RI(t)\mathrm{d}t = \mathrm{d}V_{\mathrm{noise}} = \sigma \mathrm{d}W(t)$$

where σ corresponds to a noise parameter, which is the macroscopic effect of the random electron motion.

(a) Solve this stochastic differential equation for $I(t)$, given initial condition $I(0)$.

(b) Show that, regardless of $I(0)$, the distribution of $I(t)$ as $t \to \infty$ is the same. Call a random variable with this distribution I_{eq}. You don't need to be rigorous about this step.

(c) If the circuit element is in thermal equilibrium at temperature T, and the current is given by the random variable I_{eq}, use the equipartition theorem to find an expression for σ in terms of k_{B}, T and R only.

(d) In an actual laboratory, you measure Johnson noise by hooking up a resistor to a spectral analyzer, which measures the spectral density, or Fourier transform of the autocorrelation function, of V_{noise}:

$$J(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t \, \mathrm{e}^{-\mathrm{i}\omega t} \langle V_{\mathrm{noise}}(0) V_{\mathrm{noise}}(t) \rangle.$$

The spectrum analyzer then gives you a RMS voltage, in terms of the analyzer's bandwidth ω_{B}:

$$\langle V^2 \rangle_{\mathrm{SA}} = \int_{-\omega_{\mathrm{B}}}^{\omega_{\mathrm{B}}} \mathrm{d}\omega \; J(\omega).$$

Show that

$$\langle V^2 \rangle_{\mathrm{SA}} = \frac{2}{\pi} k_{\mathrm{B}} T R \omega_{\mathrm{B}}.$$

(e) Find $\langle V^2 \rangle_{\mathrm{SA}}$ for a 100 Ω resistor at $T = 300$ K, if $\omega_{\mathrm{B}} = 1$ kHz. Is the answer reasonable?