The Distribution of Wealth

In this problem, we will determine the asymptotics of the distribution of income using a simple model. Let us suppose that an individual can have wealth $X = 0, 1, 2, \ldots$. Let us assume that this individual's wealth changes according to a continuous time birth-death process, with the rates

$$\begin{aligned} W(X = n \to X = n+1) &= \alpha + cn, \\ W(X = n \to X = n-1) &= \beta + cn \quad (n > 0). \end{aligned}$$

Of course, we need to take $\alpha < \beta$ to ensure that the distribution is well-defined! We will also take $c \ll \alpha, \beta$.

This model can be justified as follows: roughly speaking, at low amounts of wealth, where the c term is negligible, individuals essentially make random, small transactions, but they must spend money faster than they gain it (otherwise, wealth would spontaneously be generated[1]). However, when n is large, individuals are investing their wealth, and it will grow/shrink at a rate proportional to their wealth.

(a) Show that

$$\frac{\mathrm{P}_{\mathrm{eq}}(n+1)}{\mathrm{P}_{\mathrm{eq}}(n)} = \frac{\alpha + cn}{\beta + cn}.$$

(b) Show that when $n \ll \alpha/c$, $\mathrm{P}_{\mathrm{eq}}(n)$ is exponentially decaying with n.

(c) Show that when $n \gg \alpha/c$

$$\mathrm{P}_{\mathrm{eq}}(n) \sim \frac{1}{n^\gamma}$$

and find an expression for the exponent γ.

Surprisingly, both of the features of the wealth distribution argued for in this heuristic model are found in actual wealth distributions in many economies.

[1] Arguably, wealth is created in a real economy. But this arguably happens over a much longer time scale.