Traffic Jam

In this problem, we will consider a very simple model for traffic jam formation. Consider N cars which are on a circular road of length $L = bN$. Our goal will be to model the formation of a single traffic jam, which obviously may not be realistic, but will nonetheless provide some insight into the "near-critical" behavior of the model, when traffic jams just begin to form.

To begin, we need to describe the behavior of the cars. Approximate each car as a "point-like object" on the road. In a traffic jam, the cars are assumed to travel *very slowly* and to be separated by a distance $a < b$ – alternatively, we can think of this as corresponding to the length of the cars. Let us approximate that when the cars are not in a traffic jam, they travel at a speed v, which depends on the typical distance h between cars. You can approximate that the cars are evenly spaced outside of the traffic jam. The precise form of the relation between v and h we will approximate to be

$$v(h) = v_0 \frac{h^2}{h^2 + h_0^2}$$

with v_0 and h_0 positive fixed constants.

Now, we will describe the state of the traffic jam with just a single number n, the number of cars involved in the traffic jam. Since we are only considering one traffic jam forming, it is clear that the only way n can change is either that a car gets added to the back of the traffic jam, or a car accelerates away from the front. Assume that the rate at which cars leave the front of the traffic jam is

$$w_- = \frac{1}{\tau}$$

where τ is some constant; the rate at which cars enter the back of the traffic jam should be

$$w_+ = \frac{v(h)}{h}.$$

Note that since h depends on n, w_+ is an n-dependent rate.

(a) Find an "explicit" formula for the stationary distribution $\mathrm{P}_{\mathrm{eq}}(n)$ of the stochastic process.

(b) We can make a crude estimate of when a traffic jam is reasonable by asking is there an n such that $\mathrm{P}_{\mathrm{eq}}(n) < \mathrm{P}_{\mathrm{eq}}(n+1)$. Find a constraint on the variables of the problem required to satisfy this condition.

(c) Write a computer program to compute $\mathrm{P}_{\mathrm{eq}}(n)$. Show numerically that approximately when the condition of part (b), the probability of a traffic jam forming becomes non-negligible. Show some plots of $\mathrm{P}_{\mathrm{eq}}(n)$ for a reasonable value of N, and the other parameters. By reasonable, I mean a value of N where you can check that the problem is independent of the choice of N, in that the probability distribution for n/N is more or less N-independent; as for the parameters a, b, τ, v_0 and h_0, show various possible behaviors. In particular, can you find a traffic jam for which not all cars are involved (the limit $\tau \to \infty$ trivially results in a traffic jam with $n \to N$)?