3D Bosonic Strings

In this problem, we will show that fields of arbitrary spin can arise out of 3D bosonic "string theory". We will not consider interactions, but merely the light-cone quantization of a *closed* bosonic string in 3D. We've already done a good deal of the work when quantizing the bosonic string in generic dimensions, but there are some features which are special to 3D.

(a) It will be convenient for us to work with the Hamiltonian formulation of the Nambu-Goto action. Show that the action

$$S = \int \mathrm{d}\tau \oint \frac{\mathrm{d}\sigma}{2\pi} \left[P_\mu \partial_\tau X^\mu - \beta P_\mu \partial_\sigma X^\mu - \frac{\alpha}{2}\left(P^\mu P_\mu + T^2 \partial_\sigma X^\mu \partial_\sigma X_\mu\right)\right]$$

with Lagrange multipliers α and β, and momentum P^μ, is equivalent to the standard Nambu-Goto action. This form will be more convenient for us, however.

Now, let us evaluate this action using the light-cone gauge. We will define $\sqrt{2}X^\pm = X^1 \pm X^0$ and $\sqrt{2}P_\pm = P_1 \pm P_0$ (note the lowered indices) and choose the gauge where $X^+ \equiv \tau$ and $P^- \equiv p_-(\tau)$. Let us also extract zero modes x^2, p_+, p_2 and x^- from X^2, P_+, P_2 and X^- respectively (so that now spatial integrals over these four fields vanish).

(b) Explain why

$$P_+ = -\frac{P_2^2 + (T\partial_\sigma X^2)^2}{2p_-}.$$

(c) Justify the statement that X^- is a Lagrange multiplier. What does this imply about the Lagrange multiplier β? Find an expression for X^- in terms of other fields.

(d) Now consider the Poincaré symmetry generators:

$$\mathcal{P}^\mu \equiv \frac{1}{2\pi}\oint \mathrm{d}\sigma P^\mu,$$

$$\mathcal{J}^\mu \equiv \frac{1}{2\pi}\oint \mathrm{d}\sigma \epsilon^{\mu\nu\rho} X_\nu P_\rho.$$

Show that

$$-\mathcal{P}^2 \equiv m^2 = \frac{1}{2\pi}\oint \mathrm{d}\sigma \left[P_2^2 + \left(T\partial_\sigma X^2\right)^2\right],$$

$$\mathcal{P}\cdot\mathcal{J} \equiv \Lambda = \frac{p_-}{2\pi}\oint \mathrm{d}\sigma \left[X^2 P_+ - X^- P_2\right]$$

and that $[m^2, \Lambda] = 0$ as quantum operators. In 3D, Λ/m^2 corresponds to the *spin* of a state. We'll use this fact at the end of the problem.

(e) Now let us define the oscillator modes

$$P_2 - T\partial_\sigma X^2 = \sqrt{2T}\sum_{n=1}^{\infty}\left[\mathrm{e}^{\mathrm{i}n\sigma}\alpha_n + \mathrm{e}^{-\mathrm{i}n\sigma}\widetilde{\alpha}_n^\dagger\right]$$

$$P_2 + T\partial_\sigma X^2 = \sqrt{2T}\sum_{n=1}^{\infty}\left[\mathrm{e}^{-\mathrm{i}n\sigma}\alpha_n^\dagger + \mathrm{e}^{\mathrm{i}n\sigma}\widetilde{\alpha}_n\right]$$

Verify using the canonical commutation relations that $[\alpha_n, \alpha_n^\dagger] = [\widetilde{\alpha}_n, \widetilde{\alpha}_n^\dagger] = n$.

(f) Show that if we define $N = \sum \alpha_n^\dagger \alpha_n$, $\widetilde{N} = \sum \widetilde{\alpha}_n^\dagger \widetilde{\alpha}_n$, that the Lagrangian imposes a level matching condition $N = \widetilde{N}$.

(g) Conclude that up to a normal ordering ambiguity:

$$m^2 = 4T(N - a).$$

The constant a is real (but constrained, as you will see later) and reflects the normal ordering ambiguity.

(h) Next, show that $\Lambda = \sqrt{2T}(\lambda + \widetilde{\lambda})$, where

$$\lambda \equiv \sum_{n=1}^{\infty} \frac{\mathrm{i}}{n} \left(\alpha_n^\dagger \beta_n - \alpha_n \beta_n^\dagger \right)$$

and

$$\beta_n \equiv \frac{1}{2} \sum_{m=1}^{n-1} \alpha_m \alpha_{n-m} + \sum_{n>m} \alpha_m \alpha_{n-m}^\dagger$$

and $\widetilde{\lambda}$ is defined similarly.

(i) Determine the possible values of spin at levels $N = 0, 1, 2$ depending on the choice of a. Show that at level $N = 2$ by a judicious choice of a one can obtain anyons (particles where the spin is not a half-integer or integer).

It turns out that at level $N = 3$, you will always have an anyon if you did not have one at $N = 2$, for the bosonic string. This is a bizarre effect which is somewhat alleviated by supersymmetry, but not entirely – in fact, one finds a spin $1/4$ particle in $\mathcal{N} = 1$ supersymmetric 3D string theory.