The Hagedorn Temperature

In this problem, we will understand what happens to the (non-interacting) bosonic string at finite temperatures. As in quantum field theory, the proper way to understand such a theory is to consider the theory in Euclidean time with period β. In the string theory language, that corresponds to looking at strings on a Euclidean 26D space $\mathrm{S}^1 \times \mathbb{R}^{25}$. The thermal partition function $Z(\beta)$, to lowest order, will correspond to something similar to the string partition function for the compactified theory.

(a) Following the derivation of the string partition function for bosonic strings compactified on S^1, write down the thermal partition function $Z_{\mathrm{th}}(\beta)$. Exclude the sector with no momentum or winding around the circle: this is a non-thermal sector.[1]

(b) Show that

$$\int\limits_{F_0} \frac{\mathrm{d}^2\tau}{\mathrm{Im}\tau} \cdots \sum_{m,w\neq 0} \exp\left[-\frac{\beta^2}{4\pi}\frac{|m-w\tau|^2}{\mathrm{Im}\tau}\right] \sim \int\limits_{F_1} \frac{\mathrm{d}^2\tau}{\mathrm{Im}\tau} \sum_{m=1}^{\infty} \exp\left[-\frac{\beta^2}{4\pi}\frac{m^2}{\mathrm{Im}\tau}\right]$$

where F_1 is the region $\mathrm{Im}\tau > 0$, $|\mathrm{Re}\tau| < 1/2$. To do this, you will want to show that the additional modes on the left hand side can be changed by an appropriate modular transform into a pure momentum mode, at the expense of widening the region of integration. You may pick up an extra (finite) counting factor as well, but this is not important for what we'll be interested in.

(c) Using the behavior of the η function near the real line, show that $Z_{\mathrm{th}}(\beta)$ is divergent as one approaches $\beta \to \beta_{\mathrm{H}}$ from above, where

$$\beta_{\mathrm{H}} \equiv 4\pi$$

is the inverse **Hagedorn temperature**. This is the hallmark of a phase transition.

(d) Given the answers above, we may compute $F(\beta) = -\log Z_{\mathrm{th}}(\beta) + F_0$, with F_0 a "vacuum" free energy. Although $Z_{\mathrm{th}}(\beta)$ is formally divergent, we can use T duality to explore the very large temperature regime. Show that T duality implies that for very large temperatures ($\beta \to 0$):

$$F(\beta) \approx \frac{\beta_{\mathrm{H}}^2 F_0}{4\beta^2}.$$

(e) Recall that the free energy at large temperatures is related to the number of degrees of freedom. Making an analogy to quantum field theory, how many dimensions does it seem as though the high temperature string theory lives in?

We don't actually really know what is going on at high temperatures – the answer to part (e) is just a strange clue.

[1] It would also ruin our answer, due to the tachyonic divergence of the string partition function.