Hermitian Random Matrices

In many areas of quantum physics, from quantum chaos to nuclear physics, the Hamiltonian is some incredibly complicated object which we likely have no access to. Often times, one of the best approximations we can make is that the Hamiltonian H is a *random* Hermitian matrix. In this problem, we will study a Gaussian ensemble of random $N\times N$ Hermitian matrices: i.e., we assume the Hamiltonian H is drawn from a probability density

$$\mathrm{p}\left(\left(\begin{array}{ccc} H_{11} & \cdots & H_{1N}+\mathrm{i}H'_{1N} \\ \vdots & \ddots & \vdots \\ H_{1N}-\mathrm{i}H'_{1N} & \cdots & H_{NN}\end{array}\right)\right) \sim \exp\left[-\sum_i \frac{H_{ii}^2}{2\sigma^2} - \sum_{i\neq j}\frac{H_{ij}^2+H'^2_{ij}}{\sigma^2}\right]$$

For this whole problem, assume that $N \gg 1$.

(a) To begin, write some code which will generate these random matrices. Generate a large number of random matrices, of decent size, and compute the eigenvalues. Plot the resulting distribution of these eigenvalues, for a couple values of N.[1]

You should notice something strange in your results. Now, we come to the real point of this problem: we can in fact compute this eigenvalue probability distribution using a supersymmetric integral. Recall that the density of states is given by

$$\rho(E) = -\frac{1}{\pi}\mathrm{Im}\ \mathrm{tr}\frac{1}{E+\mathrm{i}0^+ - H}.$$

What we will show is that the trace can be found quite easily, in principle, from a supersymmetric integral. Actually evaluating this integral is quite challenging but it can be done asymptotically in N. In the steps below, you may neglect constant factors such as $(2\pi)^N$ which result from Gaussian integration, for simplicity, as these can easily be absorbed into the definitions of integral measures.

(b) Let z_i, $\overline{z}_i$ be complex scalars and ψ_i, $\overline{\psi}_i$ be Grassmann scalars ($i = 1,\ldots,N$). Show that

$$\mathrm{tr}\frac{1}{E-H} = \int \mathrm{d}\overline{\psi}\mathrm{d}\psi\mathrm{d}\overline{z}\mathrm{d}z\ \overline{z}_i z_i \exp\left[\mathrm{i}(E-H)_{ij}(\overline{z}_i z_j + \overline{\psi}_i\psi_j)\right].$$

(c) In the form of the integral above, it is easy to average over the random matrices H. Perform this average. Defining the graded matrix

$$Z \equiv \begin{pmatrix} \overline{z}_i z_i & \overline{\psi}_i z_i \\ \psi_i \overline{z}_i & \overline{\psi}_i\psi_i \end{pmatrix}$$

show that

$$\left\langle \mathrm{tr}\frac{1}{E-H}\right\rangle = \int \mathrm{d}\overline{\psi}\mathrm{d}\psi\mathrm{d}\overline{z}\mathrm{d}z\ \overline{z}_i z_i \exp\left[\mathrm{i}(E-H)_{ij}(\overline{z}_i z_j + \overline{\psi}_i\psi_j) - \frac{\sigma^2}{2}\mathrm{str}\left(Z^2\right)\right].$$

[1] You don't need to write any specialized code for this. When your computer can't handle it, you are working with matrices that are too large.

(d) Let X be the graded matrix

$$X = \begin{pmatrix} x & \overline{\zeta} \\ \zeta & \mathrm{i}y \end{pmatrix}.$$

Verify the following graded matrix Hubbard-Stratonovich transformation:

$$\int \mathrm{d}X \exp\left[-\frac{\mathrm{str}\left(X^2\right)}{2\sigma^2} + \mathrm{i}\,\mathrm{str}(XZ)\right] = \exp\left[-\frac{\sigma^2}{2}\mathrm{str}\left(Z^2\right)\right]$$

(e) Apply the Hubbard-Stratonovich transform to the result of part (c). Integrate by parts so that all dependence on $\overline{z}_i$ and z_i is in the exponential, and then perform the integrals over $\overline{z}_i$, z_i, $\overline{\psi}_i$, ψ_i, $\overline{\zeta}$ and ζ, to show that

$$\left\langle \mathrm{tr}\frac{1}{E-H}\right\rangle \sim \int \mathrm{d}x\mathrm{d}y\; \mathrm{e}^{-(x^2+y^2)/2\sigma^2}\left(\frac{E-\mathrm{i}y}{E+x}\right)^N\left(\frac{1}{\sigma^2} - \frac{N}{(E-\mathrm{i}y)(E+x)}\right)$$

(f) Asymptotically evaluate the integral of part (e) in the limit $N \gg 1$, keeping only the largest terms.

(g) Conclude that

$$\rho(E) = \frac{\Theta(2\sigma\sqrt{N}-E)\sqrt{4N\sigma^2 - E^2}}{2\pi N\sigma^2}.$$

Compare this answer to what you found in part (a).

What you have derived is a specific example of more general universality called **Wigner's semicircle law**, which says that the eigenvalue distribution of a random matrix is asymptotically a semicircle. This result does not depend on the details of the random matrix ensemble.