Supersymmetry and Euler Characteristics

The Euler characteristic of a 2D polygon P can be defined as

$$\chi(P) = \text{number of vertices} - \text{number of edges} + \text{number of faces}.$$

In general, there is a topological definition for the Euler characteristic corresponding to homology groups, but suffice it to say that the Euler characteristic is a very useful topological invariant for a manifold M. Some examples are for the 2-sphere or any ordinary polygon, $\chi = 2$, and for the 2-torus, where $\chi = 0$.

In this problem, we will argue that one can obtain the Euler characteristic by a very simple supersymmetric ordinary integral. To do this, we will exploit a result from Morse theory (an area of mathematics) which says that for an arbitrary manifold M, if we choose any smooth function $h : M \to \mathbb{R}$ such that at points when $\partial_i h(x) = 0$, $\det(\partial_i\partial_j h) \neq 0$, then the Euler characteristic is given by

$$\chi(M) = \sum_{x\,:\,\partial_i h(x)=0} \operatorname{sign}(\det(\partial_i\partial_j h(x))).$$

Assume that $h(x)$ has a countable set of extrema. As the formula above suggests, the right hand side is actually independent of the choice of h.

Now, how do we do this with a supersymmetric integral. Let x_i denote coordinates on M, which can be defined at least locally, and define the superfield

$$X_i = x_i + \overline{\theta}\psi_i + \overline{\psi}_i\theta + \frac{1}{2}\overline{\theta}\theta F_i$$

where $\overline{\theta}$ and θ are scalar Grassmann variables, x_i and F_i are real vectors, and ψ_i is a Grassmann vector. We then define the integral

$$Z \equiv \int \mathrm{d}X \exp\left[\int \mathrm{d}^2\theta\ h(X)\right]$$

where $h(x)$ is any smooth function satisfying the previous criteria.

(a) Show that $Z = \chi$. Thus as promised, we can compute the topological Euler characteristic with a very simple integral.

(b) To make sense of the mathematical trick, compute the Euler characteristic for the circle, S^1. Explicitly prove that it is independent of the choice of h.

Interestingly, it turns out that the Euler characteristic can be thought of as a weighted partition function (Witten index) of a supersymmetric nonlinear σ model on M. Essentially, the way this fact is proven is precisely the computation above, but of course in a more complicated setting.