Supersymmetry and Spin Glass Complexity

A typical spin glass is a model with a "mean field" energy functional of a large collection of N spins $E(s_1, \ldots, s_N)$ with an exponential number of equilibria where $\partial_{s_i} E = 0$.[1] We can characterize this by simply counting the number of solutions at energy density $\epsilon = E/N$:

$$n(\epsilon) \equiv \sum_\alpha \delta(E_\alpha - N\epsilon).$$

Here each α labels an equilibrium of the energy E.

The reason there are typically an exponential number of equilibria to E is that we actually think of E as a *random variable*, drawn from a probability distribution of disorder: for example, we may place random magnetic fields on each spin site. Spin glasses have the remarkable property that they *self-average*, and almost always the macroscopic behavior is only dependent on the distribution of the disorder, and not on the specific realization. Thus, to talk about the glass we can average over this disorder. In this spirit, let us define another variable, called the **spin glass complexity**, by

$$\exp[N\Sigma(\epsilon)] \equiv \langle n(\epsilon)\rangle_{\text{disorder}}.$$

As in usual statistical mechanics, we learn things about spin glasses through studying the partition function. At low temperatures (inverse temperature is β) we may approximate the partition function as

$$Z \approx \sum_\alpha \mathrm{e}^{-\beta E_\alpha}.$$

(a) Use a saddle point approximation on Z, averaging over disorder,[2] to show that

$$\frac{1}{\beta} = \frac{\partial \Sigma(\epsilon^*)}{\partial \epsilon},$$
$$f = \epsilon^* - \frac{\Sigma(\epsilon^*)}{\beta}.$$

Here f is the free energy per site, averaged over the disorder, and ϵ^* is the self-averaged average energy per site – it is by far the most likely energy we would see on a given realization of the spin glass. Comment on these results – their form should look familiar, although they are appearing in a slightly different context.

Now, where does supersymmetry come in to this picture? We can also count states by a supersymmetric "path" integral. I'll get you started: consider the integral

$$n(\epsilon) = \sum_\alpha \int \mathrm{d}s_i \; \delta(s_i - s_{i,\alpha})\delta(E_\alpha - N\epsilon).$$

Here $s_{i,\alpha}$ is the value of spin s_i in equilibrium α.

[1] Our function E is often referred to as the TAP free energy.

[2] Actually, we almost always want to average over the disorder on the free energy, not the partition function. This is a subtlety which you don't need to worry about in this problem. You will see the interesting physics regardless.

(b) Show that

$$n(\epsilon) = \int \mathrm{d}s_i \mathrm{d}\lambda_i \mathrm{d}\mu \mathrm{d}\bar{c}_i \mathrm{d}c_i \; \exp\left[\mathrm{i}\mu(E(s) - N\epsilon) + \mathrm{i}\lambda_i \frac{\partial E(s)}{\partial s_i} + \bar{c}_i c_j \frac{\partial^2 E(s)}{\partial s_i \partial s_j}\right]$$

where $\bar{c}_i$, c_i are Grassmann variables, and λ_i and μ are real.

(c) Show that the integral of part (b) has a BRST supersymmetry.

(d) Find an operator whose BRST variation provides the Ward identity

$$0 = \langle s_i \lambda_j + \bar{c}_j c_i \rangle$$

and be sure to note over what probability distribution we are averaging over.

(e) Argue that, if h_j corresponds to a magnetic field on site j, the result of part (d) leads to

$$\left.\frac{\mathrm{d}\langle s_i \rangle}{\mathrm{d}h_j}\right|_{h_j=0} = \left\langle \left(\frac{\partial^2 E(s)}{\partial s_i \partial s_j}\right)^{-1} \right\rangle$$

which is the **static fluctuation-dissipation theorem**.

We have shown above how the static fluctuation-dissipation theorem, which relates energy curvature to the response of the system under external perturbations, can be thought of as a trivial Ward identity from a supersymmetric path integral. But the story gets a bit weird now. Another feature of spin glasses is that the spectrum of the equilibria is *very sensitive* to external parameters. In particular, as we tune up a magnetic field h_j, we may destroy and create new equilibria so fast that what we mean by the distribution we averaged over in parts (d-e) changes between 0 and $\mathrm{d}h_j$. In particular, this can mean that, roughly speaking:

$$\left.\frac{\mathrm{d}\langle s_i \rangle}{\mathrm{d}h_j}\right|_{h_j=0} \neq \left.\left\langle \frac{\mathrm{d}s_i}{\mathrm{d}h_j} \right\rangle\right|_{h_j=0} = \left\langle \left(\frac{\partial^2 E(s)}{\partial s_i \partial s_j}\right)^{-1} \right\rangle$$

This violation of the BRST Ward identity must thus be interpreted as *spontaneous supersymmetry breaking* in the state-counting integral. This is a bit more than just a theoretical conjecture – there is substantial evidence that this in fact happens for some famous spin glass models.