Calabi-Yau Elliptic Fibrations over S^2s

Consider an elliptic fibration over the space $(\mathrm{S}^2)^n$ given by

$$y^2 = x^3 + f(z_1, \ldots, z_n)x + g(z_1, \ldots, z_n)$$

where $z_1, \ldots, z_n$ describe complex coordinates on each sphere. Assume that f and g are polynomials, and let us assume that this polynomial equation has 3 unique zeroes.

(a) Show that if this fibration is to be a Calabi-Yau $(n+1)$-fold, the largest powers of each z_i in the polynomials f and g are uniquely fixed. Do this by requiring that there be a globally defined holomorphic $(n+1)$-form with no zeroes or poles.

(b) How many complex structure moduli can you identify? Be careful not to overcount! Guess at some of the Hodge numbers of this Calabi-Yau.

(c) How many Kähler moduli can you identify? Guess at more of the Hodge numbers of this Calabi-Yau.

(d) In the case of $n = 1$, this is a description of some K3 spaces. Comment on the Hodge numbers of K3, compared with what you found above.

In reality, of course, there are always points at which two of the zeroes of f and g align, corresponding to singular torus fibers. In general, these do not correspond to singularities of the Calabi-Yau, but rather "bad fibrations" near these points.