Hermitian Matrices and Symplectic Manifolds

Let $\mathcal{H}$ be the set of $n \times n$ Hermitian matrices. There is a natural action on $\mathcal{H}$ of $\mathrm{U}(n)$ given by conjugation: for $R \in \mathcal{H}$ and $V \in \mathrm{U}(n)$:

$$R \to URU^{-1}.$$

Let us denote by $\mathcal{H}_{(\lambda_1,\ldots,\lambda_n)}$ the subspace of $\mathcal{H}$ consisting of matrices whose eigenvalues are $\lambda_1, \ldots, \lambda_n$. We will usually just denote this space by $\mathcal{H}_\lambda$, for simplicity. Note that we may have multiplicities, e.g. $\lambda_1 = \lambda_2$. In this problem, we will show that $\mathcal{H}_\lambda$ are symplectic manifolds, and explore some interesting consequences of this.

(a) Show that the orbits of the conjugation action on $\mathcal{H}$ are the $\mathcal{H}_\lambda$.

(b) Note that $\mathrm{i}\mathcal{H} = \mathrm{u}(n)$. For $R \in \mathcal{H}_\lambda$, let us define a 2 form, acting on a pair of vectors $X, Y \in \mathrm{u}(n)$:

$$\omega_R(X, Y) = \mathrm{i}\,\mathrm{tr}([X, Y]R).$$

Show that $\mathrm{Ker}(\omega_R)$ corresponds to the Lie algebra of the stabilizer of $\mathcal{H}_\lambda$ under the $\mathrm{U}(n)$ action. What is this Lie algebra, explicitly?

(c) Show that $(\mathcal{H}_\lambda, \omega_R)$ is a symplectic manifold.

(d) Suppose that $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$. Describe the orbit $\mathcal{H}_\lambda$ by giving a simple geometric picture. Is the space the same for all λ?

(e) Show that if $\lambda_1 = a$ and $\lambda_i = b \neq a$, for all $i \neq 1$, then the orbit $\mathcal{H}_\lambda = \mathbb{C}\mathrm{P}^{n-1}$. This provides a rather direct proof that $\mathbb{C}\mathrm{P}^{n-1}$ is a symplectic manifold.