Stiefel Manifolds

The Stiefel manifold $V_m(\mathbb{R}^n)$ is defined as the set of all ordered pairs of m orthonormal vectors in $\mathbb{R}^n$: in symbols, that means

$$V_m(\mathbb{R}^n) \equiv \{(e_1, \ldots, e_m) \in \mathbb{R}^n \;:\; e_i \cdot e_j = \delta_{ij}\}.$$

Note that you can think of $(e_1, \ldots, e_m)$ as a matrix in $\mathbb{R}^{n\times m}$: this is sometimes a convenient way to think of points in the manifold.

(a) Show that $V_1(\mathbb{R}^n) = \mathrm{S}^{n-1}$. Thus, Stiefel manifolds are, in a sense, generalizations of spheres.

(b) Show that $\mathrm{SO}(n)$ acts transitively on $V_m(\mathbb{R}^n)$.

(c) Find the isotropy group of any point in $V_m(\mathbb{R}^n)$.

(d) Conclude that $V_m(\mathbb{R}^n) = \mathrm{SO}(n)/\mathrm{SO}(n-m)$.

(e) What is $\dim(V_m(\mathbb{R}^n))$?