The Homology of $\mathbb{RP}^n$

Real projective spaces come up frequently in physical applications. They are defined as follows:

$$\mathbb{RP}^n \equiv (\mathbb{R}^{n+1} - \{0\})/\sim, \quad \text{where } x \sim \lambda x \text{ for any } \lambda \in \mathbb{R},\ \lambda \neq 0.$$

They come up in physics for a variety of reasons. A classical example is in the study of liquid crystals, which are long rod-shaped molecules. Roughly speaking, these molecules are symmetric, therefore there is no physical distinction between the states of the molecule if you invert it around the center of the rod. Therefore, one can describe the orientation of the molecule as a line in $\mathbb{R}^3$, passing through a fixed point (the origin). The space of all such lines passing through the origin is, as defined above, $\mathbb{RP}^2$.

In this problem, we will find a description of $\mathbb{RP}^n$ as a CW complex, for each n. It is then easy to use this CW complex to compute the homology groups of these spaces.

(a) Show that an alternative description of $\mathbb{RP}^n$ is $\mathrm{S}^n/\sim$, where $\sim$ is the equivalence relation that $p \sim -p$ for all points p on S^n.

(b) We may now exploit the fact that S^n is the universal cover of $\mathbb{RP}^n$ (for $n > 1$) to find an intuitive CW structure for $\mathbb{RP}^n$. First, construct a CW complex for S^n by using 2 k cells, for every $0 \leq k \leq n$. This can be done recursively, where you think of $\mathrm{S}^{n-1} \subset \mathrm{S}^n$ as an equator of the sphere.

(c) Combine the previous two parts to find a CW structure for $\mathbb{RP}^n$.

(d) Conclude that

$$\begin{aligned}
&\mathrm{H}_0(\mathbb{RP}^n) = \mathbb{Z}, \\
&\mathrm{H}_k(\mathbb{RP}^n) = \begin{cases} 0 & k \text{ even} \\ \mathbb{Z}_2 & k \text{ odd} \end{cases} \quad (0 < k < n), \\
&\mathrm{H}_n(\mathbb{RP}^n) = \begin{cases} 0 & n \text{ even} \\ \mathbb{Z} & n \text{ odd} \end{cases}, \\
&\mathrm{H}_k(\mathbb{RP}^n) = 0 \quad (k > n).
\end{aligned}$$