The Homology of $\mathrm{U}(2)$

The Lie group $\mathrm{U}(2)$ can be defined in terms of matrices in $\mathbb{C}^{2\times 2}$:

$$\mathrm{U}(2) \equiv \{A \in \mathbb{C}^{2\times 2} \ : \ A^\dagger A = 1\}.$$

The goal of this problem will be to compute the homology groups for this Lie group. In general, this is a very challenging task, but for this group in particular it will be doable!

We'll first need to show the following preliminary results.

(a) Show that, for some $x_0 \in \mathrm{S}^n$,

$$\mathrm{H}_i(X \times \mathrm{S}^n) = \mathrm{H}_i(X) \oplus \mathrm{H}_i(X \times \mathrm{S}^n, X \times \{x_0\}).$$

(b) Show that

$$\mathrm{H}_i(X \times \mathrm{S}^n, X \times \{x_0\}) \cong \mathrm{H}_{i-1}(X \times \mathrm{S}^{n-1}, X \times \{x_0\}).$$

(c) Conclude that, if we take $\mathrm{H}_j(X) = 0$ for $j < 0$:

$$\mathrm{H}_i(X \times \mathrm{S}^n) \cong \mathrm{H}_i(X) \oplus \mathrm{H}_{i-n}(X).$$

Now, let's turn to $\mathrm{U}(2)$.

(d) Show that $\mathrm{U}(2) = \mathrm{SU}(2) \times \mathrm{S}^1$, where

$$\mathrm{SU}(2) \equiv \{A \in \mathrm{U}(2) \ : \ \det(A) = 1\}.$$

(e) Construct an explicit homeomorphism between $\mathrm{SU}(2)$ and S^3.

(f) Using the previous parts, find the homology groups of $\mathrm{U}(2)$.