Uniqueness of Virasoro Algebra

Due to normal ordering ambiguities, we expect the most general possible form of the quantum Virasoro algebra to be

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m,-n}.$$

(a) Show that by shifting L_0 by a constant (which won't change the algebra) that one can always arrange for $A(1) = 0$.

(b) Show that if $A(1) = 0$, then L_0 and $L_{\pm}1$ form a closed algebra isomorphic to $\mathrm{so}(2,1)$.

(c) Show that the Jacobi identity

$$[[L_m, L_n], L_p] + [[L_n, L_p], L_{,}m] + [[L_p, L_m], L_n] = 0$$

implies that

$$A(m) = \frac{c}{12}\left(m^3 - m\right)$$

with c a constant, which we have called the central charge.

The added term is often called the **central extension** of the Virasoro algebra.